APPENDIX 21A: Conditional EPE calculation and WWR

More details of the calculations below can be found in Rosen and Pykhtin (2010).

i) EPE for a forward contract under the assumption of a normally distributed value

In Appendix 11B, we derived a simple formula for the expected exposure (EPE) for an underlying value (V_t) of the form:

$$dV_t = \mu dt + \sigma dW_t,$$

where μ represents a drift and σ is a volatility of the exposure with dW_t representing a standard Brownian motion. The EPE for a given time horizon s is given by:

$$EPE_s = \mu s \Phi\left(\frac{\mu}{\sigma}\sqrt{s}\right) + \sigma s \varphi\left(\frac{\mu}{\sigma}\sqrt{s}\right).$$

ii) EPE expression conditional on default

Now we derive a similar formula but conditioning on some actual default time. Under the above assumptions, the value of a contract at some time s in the future is given by:

$$V(s) = \mu s + \sigma\sqrt{s}Y,$$

where Y is a Gaussian random variable. Let us denote the time of default of the counterparty by τ and the default probability of the counterparty up to time s as $F(s)$ which, as in Appendix 12A, is defined via a constant hazard rate h or intensity of default:

$$F(s) = 1 - \exp(-hs)$$

Like the exposure, default is driven by a Gaussian variable, Z:

$$\tau = F^{-1}(\Phi(Z))$$

Finally, we link the Gaussian variables Y and Z via a correlation parameter ρ:

$$Y = \rho Z + \sqrt{1-\rho^2}\varepsilon,$$

with ε being a further (independent) Gaussian variable. We now need to calculate the EPE conditional upon default having occurred. This is:

$$EPE(s|\tau = s) = E[\max(V(s), 0)|Z = \Phi^{-1}(F(\tau))]$$

$$= \int_{-\mu(t)/\sigma(t)}^{\infty} [\mu'(s) + \sigma'(s)]\varphi(x)dx$$

Where

$$\mu'(s) = \mu(s) - \rho\sigma\Phi^{-1}(F(\tau)) \qquad \sigma'(s) = \sqrt{1-\rho^2}\sigma$$

The conditional EPE is then give by:

$$EPE(s|\tau = s) = \mu'(s)\Phi\left(\frac{\mu'(s)}{\sigma'(s)}\right) + \sigma'(s)\varphi\left(\frac{\mu'(s)}{\sigma'(s)}\right).$$

This is illustrated in Spreadsheet 21.1.

APPENDIX 21B: Survival probabilities in the CIR model

In the standard CIR model, the intensities are represented by:

$$d\lambda_t = k(\theta - \lambda_t)dt + \sigma\sqrt{\lambda_t}dW_t$$

The survival probabilities are analogous to the discount factors in the interest rate specification (Cox, Ingersoll and Ross (1985):

$$S(t,T) = A(t,T)\exp\left(-B(t,T)\,\lambda_t\right)$$

with

$$A(t,T) = \left[\frac{2h\exp\{(k+h)(T-t)/2\}}{2h+(k+h)[\exp\{(T-t)h\}-1]}\right]^{2k\theta/\sigma^2}$$

$$B(t,T) = \frac{2[\exp\{(T-t)h\}-1]}{2h+(k+h)[\exp\{(T-t)h\}-1]}$$

$$h = \sqrt{k^2+2\sigma^2}$$

When adding jumps, the term for $A(t,T)$ above needs to be multiplied by the following factor:

$$\left[\frac{2h\exp\{(k+h+2\gamma)(T-t)/2\}}{2h+(k+h+2\gamma)[\exp\{(T-t)h\}-1]}\right]^{\frac{2\alpha\gamma}{\sigma^2-2k\gamma-2\gamma^2}}$$

In both cases, it is possible to introduce a deterministic shift as a function of time to exactly match the survival curve (Brigo and Mercurio 2001).

APPENDIX 21C: Simple initial margin calculation

As noted in Appendix 11A, the expected positive exposure (EPE) of a normal distribution can be written as:

$$EPE = \mu\Phi\left(\frac{\mu}{\sigma}\right) + \sigma\varphi\left(\frac{\mu}{\sigma}\right)$$

For the collateralised case (zero threshold, no initial margin) the impact of the margin period of risk would lead to $\mu = 0$ and $\sigma = \sqrt{\tau_{MPR}}$ giving an expected exposure (EPE) of:

$$EPE_{no\ IM} = \sqrt{\tau_{MPR}}\varphi(0) = \sqrt{\tau_{MPR}}(2\pi)^{-0.5}$$

The impact of initial margin can be considered equivalent to shifting the mean of the distribution to be $\mu = -\Phi^{-1}(\alpha)\sqrt{\tau_{IM}}$ where τ_{IM} is the time horizon and α the confidence level used to define the initial margin (the initial margin is assumed to be also calculated from normal distribution assumptions potentially using a different time horizon). This leads to an EPE of:

$$EPE_{IM} = -\Phi^{-1}(\alpha)\sqrt{\tau_{IM}}\Phi\left(\frac{-\Phi^{-1}(\alpha)\sqrt{\tau_{IM}}}{\sqrt{\tau_{MPR}}}\right) + \sqrt{\tau_{MPR}}\varphi\left(\frac{-\Phi^{-1}(\alpha)\sqrt{\tau_{IM}}}{\sqrt{\tau_{MPR}}}\right)$$

This can be simplified to give:

$$EPE_{IM} = \sqrt{\tau_{MPR}}\varphi\left(\sqrt{\lambda}K\right) - K\sqrt{\tau_{IM}}\Phi\left(-\sqrt{\lambda}K\right)$$

where $\lambda = \tau_{IM}/\tau_{MPR}$ is the ratio of the time horizon used (τ_{IM}) for the IM calculation divided by the MPR for the exposure quantification (τ_{MPR}) and $K = \Phi^{-1}(\alpha)$ where $\varphi(.)$ is a standard normal density function and $\Phi(.)$ is the cumulative standard normal density function.

Finally:

$$R_\alpha = \frac{EPE_{no\ IM}}{EPE_{IM}} = \left[\varphi\left(\sqrt{\lambda}K\right) - K\sqrt{\lambda}\Phi\left(-\sqrt{\lambda}K\right)\right]^{-1}(2\pi)^{-0.5}.$$

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