APPENDIX 19A: Optimal hedging and beta

In Appendix 14E, we derived the optimal beta for hedging a single asset or liability (DVA in the case considered) with a CDS position. The optimal beta is:

$$\beta = -\rho_{AB}\frac{\sigma_A}{\sigma_B}.$$

With the overall variance being

$$\sigma_A^2(1-\rho_{AB}^2).$$

The optimal hedge therefore leads to a smaller variance multiplied by the factor $(1-\rho_{AB}^2)$ or standard deviation multiplied by the factor $\sqrt{1-\rho_{AB}^2}$. This treatment is for a single underlying (based in the DVA example on credit spread exposure).

Now suppose there is a homogenous portfolio of n assets each with volatility σ_A being hedged with another single asset B. The return on asset B will be a Gaussian variable V_B whilst the return of each of the assets in the portfolio is:

$$V_i = \rho V_B + \sqrt{1-\rho^2}V_i'$$

where the V_i' are uncorrelated Gaussian variables. The correlation of each asset in the portfolio to the asset B is ρ whilst the correlation between different assets is ρ^2. The total change in portfolio value is:

$$\sum_i \sigma_A V_i + \beta\sigma_B V_B = \sum_i \sigma_A\left(\rho V_B + \sqrt{1-\rho^2}V_i'\right) + \beta\sigma_B V_B$$
$$= (n\sigma_A\rho + \beta\sigma_B)V_B + \sum_i \sigma_A\sqrt{1-\rho^2}V_i'$$

The variance is:

$$(n\sigma_A\rho + \beta\sigma_B)^2 + n(1-\rho^2)\sigma_A^2$$

The first term above can be seen as systematic risk arising from the relationship to the index and the second is idiosyncratic risk.

With no hedge ($\beta = 0$) then the variance is:

$$(n\sigma_A\rho)^2 + n(1-\rho^2)\sigma_A = \sigma_A^2(n^2\rho^2 + n - n\rho^2)$$

The variance minimising hedge is found by differentiation:

$$2\beta\sigma_B^2 + 2n\sigma_A\sigma_B\rho = 0$$
$$\beta = -n\rho\sigma_A/\sigma_B$$

With the variance minimising hedge then the variance is:

$$(n\sigma_A\rho - n\rho\sigma_A\sigma_B/\sigma_B)^2 + n(1-\rho^2)\sigma_A^2 = n(1-\rho^2)\sigma_A^2$$

The ratio of the variance with the optimal beta hedge divided by the variance with no hedge is therefore:

$$\frac{(1-\rho^2)}{n\rho^2 + (1-\rho^2)}$$

The hedging benefit (Figure 19.4) is the square root of this and equals the previous case for $n = 1$.

Copyright © 2026 Jon Gregory jon@oftraining.com