APPENDIX 14A: Unilateral CVA formula derivation

We wish to find an expression for the risky value, $V(t,T)$, of a netted set of derivatives positions with a maximum maturity date T. Denote the no default value of the relevant positions as $V_{ND}(t,T)$ and the (random) default time of the counterparty as τ. Then $V(s,T)$ $(t < s \leq T)$ will denote the future value including discounting.

There are two cases to consider:

1) Counterparty does not default before T.

In this case, the risky position is equivalent to the risk-free position and we write the corresponding payoff as:

$$I(\tau > T)V_{ND}(t,T)$$

where $I(\tau > T)$ is the indicator function denoting default (this takes the value 1 if default has not occurred before or at time T and zero otherwise).

2) Counterparty does default before T

In this case, the payoff consists of two terms, the value of the position that would be paid *before* the default time (all cash flows before default will still be paid by the counterparty) plus the payoff at default.

i) Cashflows paid up to the default time

$$I(\tau \leq T)V_{ND}(t,\tau)$$

ii) Default payoff

Here, if the value of the trade at the default time, $V_{ND}(\tau,T)$, is positive then the institution will receive a recovery fraction (R) of the default free value of the derivatives positions whilst if it is negative then they will still have to settle this amount. Hence, the default payoff at time τ is:

$$I(\tau \leq T)[R.V_{ND}(\tau,T)_+ + V_{ND}(\tau,T)_-]$$

where $x_- = \min(x,0)$ and $x_+ = \max(x,0)$.

Putting the above payoffs together, gives the following expression for the actual value including counterparty risk:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau > T)V_{ND}(t,T) + \\ I(\tau \leq T)V_{ND}(t,\tau) + \\ I(\tau \leq T)[R.V_{ND}(\tau,T)_+ + V_{ND}(\tau,T)_-] \end{bmatrix}$$

Re-arranging and using the relationship $x_- = x - x_+$ we obtain:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau > T)V_{ND}(t,T) + \\ I(\tau \leq T)V_{ND}(t,\tau) + \\ I(\tau \leq T)[R.V_{ND}(\tau,T)_+ V_{ND}(\tau,T) - V_{ND}(\tau,T)_+] \end{bmatrix}$$

Since $V_{ND}(t,\tau) + V_{ND}(\tau,T) = V_{ND}(t,T)$ then this becomes:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau > T)V_{ND}(t,T) + \\ I(\tau \leq T)V_{ND}(t,T) + \\ -I(\tau \leq T)(1-R)V_{ND}(\tau,T)_+ \end{bmatrix}$$

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Which simplifies to:

$$V(t,T) = V_{ND}(t,T) + E^{Q}[-I(\tau \leq T)(1-R)V_{ND}(\tau,T)_{+}]$$

Note that we made the assumption that the future default free value, $V_{ND}(\tau,T)$, includes discounting for notational simplicity and also note that we sometimes write the loss given default (LGD) instead of $(1-R)$. Equation (14.1) in the book therefore identifies the unilateral CVA (UCVA) as the right term in this formula:

$$V(t,T) = V_{ND}(t,T) - E^{Q}\left[I(\tau \leq T).LGD.[V_{ND}(\tau,T)_{+}]\right].$$

The above formula for UCVA is completely general. A common simplification is to assume that the exposure, default time and LGD are independent (no wrong-way risk, for example) which leads to the formula below (Equation 14.4 in the book):

$$= -ELGD\sum_{i=1}^{N} EPE(t_i)\left[\exp\left(-\frac{s(t_{i-1})}{ELGD}.t_{i-1}\right) - \exp\left(-\frac{s(t_i)}{ELGD}.t_i\right)\right]$$

Where assumptions must be made on the form of the credit spread $s(.)$ or alternatively the form of the default probability function as discussed in Appendix 11A.

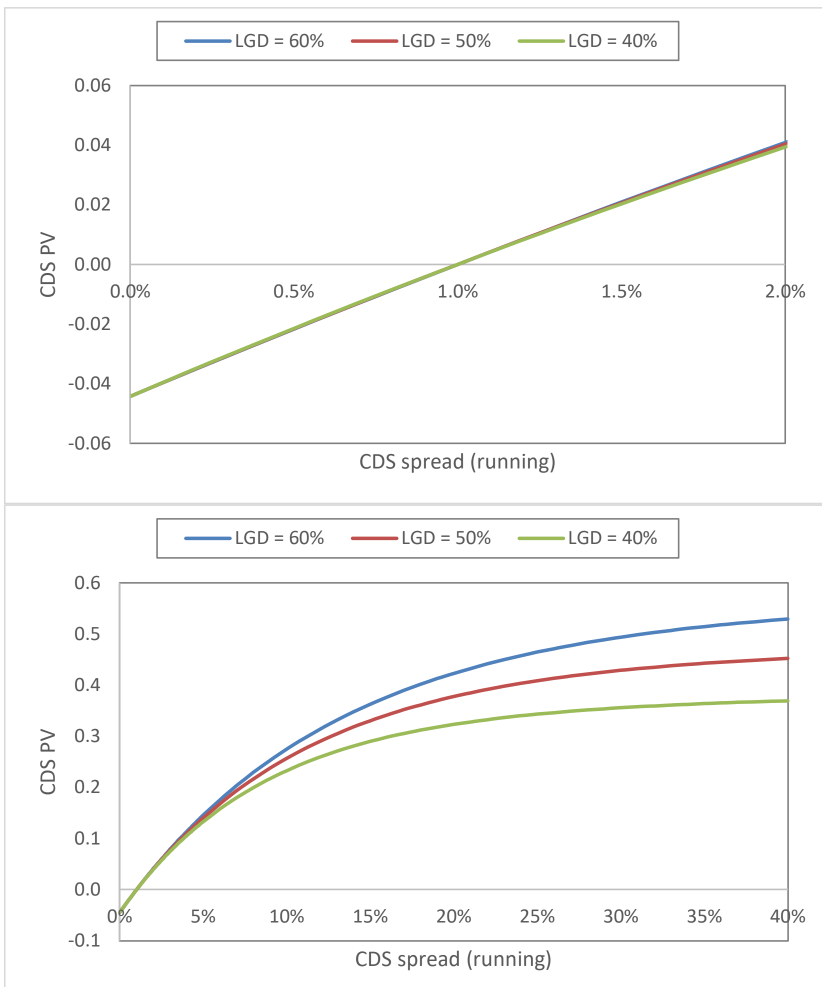
APPENDIX 14B: LGD Assumptions and Seniority

In the standard UCVA formula, it is possible to interpret the term on the far right hand side as a risk-neutral default probability:

$$= -ELGD \sum_{i=1}^{N} EPE(t_i) \left[\exp\left(-\frac{s(t_{i-1})}{ELGD} . t_{i-1}\right) - \exp\left(-\frac{s(t_i)}{ELGD} . t_i\right) \right]$$

This is also the case for the protection leg of a credit default swap (CDS) which has the same formula as the above but with the EPE replaced by a discount factor (i.e. it is the CVA on a unit exposure). For either CVA calculation or CDS valuation, the ELGD in the above formula is generally a model parameter since it is not observable (since products such as recovery swaps and digital CDS do not trade). However, the impact of ELGD is second order and very small unless the underlying credit spread is high as illustrated in Figure 1.

Figure 1. *Present value (PV) for a CDS with a premium of 1% as a function of the market CDS spread and different LGDs (ELGDs). The bottom graph shows a much larger range on the x-axis.*



The ELGD is therefore not a particularly important parameter in CDS valuation (or CVA calculations) unless the credit is particularly distressed. In such a case, CDS tend to trade on an upfront basis, revealing the implied ELGD.

It is necessary to have conventions on ELGD in the CDS market in order to agree on valuations. These conventions are based loosely around the real world empirically observed average values but are not granular. For example, the convention for corporate names is usually 60% whereas there is empirical evidence that the expected LGD of corporates depends on industry.

Now consider the valuation of a digital CDS or calculation of CVA where the LGD is contractually 100%. It would be tempting to adapt the formula as:

$$= -100\%. \sum_{i=1}^{N} EPE(t_i) \left[\exp\left(-\frac{s(t_{i-1})}{ELGD}.t_{i-1}\right) - \exp\left(-\frac{s(t_i)}{ELGD}.t_i\right)\right]$$

However, in the above formula the value of 100% is a deterministic payoff that replaces the previous $ELGD$ parameter which is stochastic and a model parameter. It would be more accurate to write:

$$= -E\left[\frac{100\%}{LGD}\right].ELGD. \sum_{i=1}^{N} EPE(t_i) \left[\exp\left(-\frac{s(t_{i-1})}{ELGD}.t_{i-1}\right) - \exp\left(-\frac{s(t_i)}{ELGD}.t_i\right)\right]$$

This would explicitly reflect the fact that the digital CDS is worth 100% divided by the realised LGD times more than the standard CDS.

The case of CVA with a derivative seniority that differs from the underlying market used to derive the credit spread is similar to the case of the digital CDS. Often the formula is modified as:

$$= -ELGD_{deriv} \sum_{i=1}^{N} EPE(t_i) \left[\exp\left(-\frac{s(t_{i-1})}{ELGD}.t_{i-1}\right) - \exp\left(-\frac{s(t_i)}{ELGD}.t_i\right)\right]$$

However, this inconsistently replaces a market convention ELGD with a value that will be presumably estimated with some data on the seniority in question. The following formula is more appropriate:

$$= -E\left[\frac{LGD_{deriv}}{LGD_{cds}}\right] ELGD \sum_{i=1}^{N} EPE(t_i) \left[\exp\left(-\frac{s(t_{i-1})}{ELGD}.t_{i-1}\right) - \exp\left(-\frac{s(t_i)}{ELGD}.t_i\right)\right]$$

The term $E[LGD_{deriv}/LGD_{cds}]$ can be considered as a seniority factor that adjusts for the relative seniority of the derivative exposure compared to that for the underlying CDS market used to estimate the credit spread curve. Note that this is not the same as $E[LGD_{deriv}]/E[LGD_{cds}]$ although data for computing this ratio of expected values may be more readily available through published studies on LGD or recovery rates (e.g. see Table 11.4 in the book). Note that the above also requires that the spread is estimated on only a probability of default basis with seniority adjustments made only to the LGD.

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APPENDIX 14C: CVA as a running spread

Please see the following chapter (which is available on www.cvacentral.com) for more details.

www.cvacentral.com/wp-content/uploads/2014/05/VrinsGregory_RunningCVA.pdf

Vrins, F., and J. Gregory, 2011, "Getting CVA up and running", Risk, October.

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APPENDIX 14D: Bilateral CVA formula (CVA and DVA)

We wish to find an expression for the risky value, $V(t,T)$, of a netted set of derivatives positions with a maximum maturity date T as in Appendix 14A but under the assumption that the party concerned may also default in addition to their counterparty. Denoting the default time of the party making the calculation as τ_I, their recovery value as R_I and following the notation and logic in Appendix 14A with the counterparty now indexed by C, we now have the following cases:

1) Neither counterparty nor institution defaults before T

In this case, the risky position is equivalent to the default-free position and we write the corresponding payoff as:

$$I(\tau^1 > T)V_{ND}(t,T)$$

where the 'first-to-default' time of the institution and counterparty is defined as $\tau^1 = \min(\tau_C, \tau_I)$.

2) Counterparty defaults first and also before time T

This is the default payoff as in Appendix 14A:

$$I(\tau^1 \leq T)I(\tau^1 = \tau_C)[R_C.V_{ND}(\tau^1,T)_+ + V_{ND}(\tau^1,T)_-]$$

3) Party making the calculation defaults first and also before time T

This is an additional term compared with the unilateral CVA case and corresponds to the institution itself defaulting. If they owe money to their counterparty (negative value) then they will pay only a recovery fraction of this whilst if the counterparty owes them money (positive value) then they will still receive this. Hence, the payoff is the opposite of case 2 above:

$$I(\tau^1 \leq T)I(\tau^1 = \tau_I)[R_I.V_{ND}(\tau^1,T)_- + V_{ND}(\tau^1,T)_+]$$

4) If either the institution or counterparty does default then all cashflows prior to the first-to-default date will be paid

$$I(\tau^1 \leq T)V(t,\tau^1)$$

Putting the above payoffs together, we have the following expression for the value including bilateral counterparty risk:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau^1 > T)V_{ND}(t,T) + \\ I(\tau^1 \leq T)V_{ND}(t,\tau^1) + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_C)[R_C.V_{ND}(\tau^1,T)_+ + V_{ND}(\tau^1,T)_-] + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_I)[R_I.V_{ND}(\tau^1,T)_- + V_{ND}(\tau^1,T)_+] \end{bmatrix}$$

Similarly to Appendix 14A, we can simplify the above expression as:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau^1 > T)V_{ND}(t,T) + \\ I(\tau^1 \leq T)V_{ND}(t,\tau^1) + I(\tau^1 \leq T)V_{ND}(\tau^1,T) + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_C)[R_C.V_{ND}(\tau^1,T)_+ - V_{ND}(\tau^1,T)_+] + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_I)[R_I.V_{ND}(\tau^1,T)_- - V_{ND}(\tau^1,T)_-] \end{bmatrix}$$

Finally obtaining:

$$V(t,T) = V_{ND}(t,T) + E^Q \begin{bmatrix} I(\tau^1 \le T)I(\tau^1 = \tau_C)[R_C.V_{ND}(\tau^1,T)_+ - V_{ND}(\tau^1,T)_+] + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)[R_I.V_{ND}(\tau^1,T)_- - V_{ND}(\tau^1,T)_-] \end{bmatrix}$$

The terms on the right hand side can be identified as the CVA and DVA such as in Equations 14.8b and 14.8c in the book. Note that to separate the CVA and DVA terms we must assume independence between the default times τ_C and τ_I as well as making the other standard assumptions of independence and no wrong-way risk. Note also that the terms above are contingent (as in Equations 14.8b and 14.8c) and imply the use of survival probabilities (see discussion in Section 14.3.2).

APPENDIX 14E: DVA Hedging

Suppose that we hedge an asset A with another asset B. This can have many applications such as considering the risk of offsetting equity positions (long one equity and short another, for example) but we are interesting in the implication on hedging DVA with the CDS on a credit spread that is correlated to that used in calculating the DVA (for example, a bank may hedge their DVA by selling CDS protection on another bank). This problem is also sometimes referred to as beta-hedging.

Assuming a unit size of asset A (the DVA) and a quantity β of asset B (the CDS hedging position), the overall variance of the position will be:

$$\sigma_A^2 + \beta^2\sigma_B^2 + 2\beta\rho_{AB}\sigma_A\sigma_B$$

where ρ_{AB} is the correlation between A and B and σ represents standard deviation. We could choose the value of β by minimising the variance:

$$2\beta\sigma_B^2 + 2\sigma_A\sigma_B = 0$$

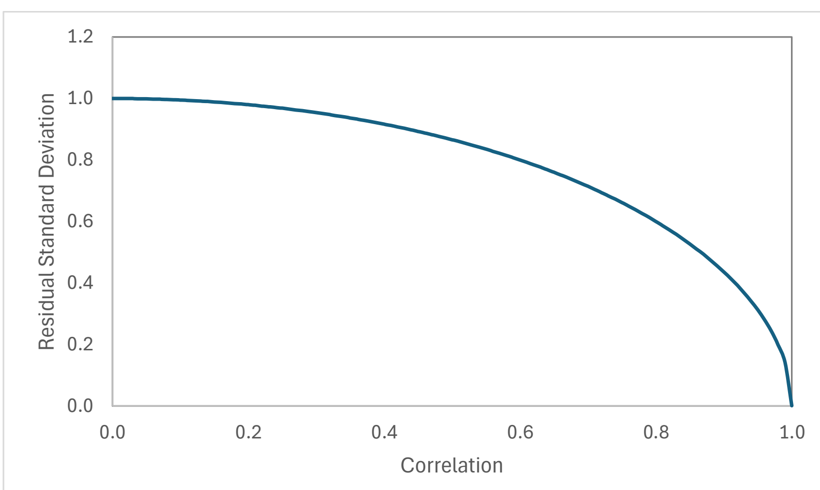
$$\beta = -\rho_{AB}\frac{\sigma_A}{\sigma_B}$$

The optimal hedge is therefore proportional to the correlation and of the opposite sign (e.g. being short credit risk through DVA means being long credit risk on the hedge). Finally, substituting the optimal beta to find the variance:

$$\sigma_A^2 + \rho_{AB}^2\frac{\sigma_A^2}{\sigma_B^2}\sigma_B^2 - 2\rho_{AB}\frac{\sigma_A}{\sigma_B}\rho_{AB}\sigma_A\sigma_B = \sigma_A^2 + \rho_{AB}^2\sigma_A^2 - 2\rho_{AB}^2\sigma_A^2$$

$$= \sigma_A^2 - \rho_{AB}^2\sigma_A^2 = \sigma_A^2\,(1 - \rho_{AB}^2)$$

The standard deviation from an optimal beta hedge is therefore $\sqrt{1-\rho_{AB}^2}$ times the original standard deviation (see graph below). For a 50% effective hedge, we would need the correlation to be around 87%. This implies DVA hedging is problematic unless a consistency and highly correlation proxy is available.



APPENDIX 14F: Collateralised EPE approximation

It is interesting to assess the reduction of EPE (or ENE) due to a collateral agreement as a function of the margin period of risk (MPoR) and maturity of the underlying portfolio. Since CVA (DVA) is approximately proportional to the EPE (ENE) then this same reduction can be used to assess the likely impact on CVA (or DVA). The broad assumptions in deriving this formula are strong collateralisation (zero threshold but no initial margin).

i) Uncollateralised case

As discussed in Appendix 10B, a simple proxy for the standard deviation of an uncollateralised portfolio is $\sigma\sqrt{t}(T-t)$ where T is the longest maturity in the portfolio and σ is some volatility term (for example for an interest swap portfolio this would be approximately some weighted average interest rate volatility). Under normal distribution assumptions, assuming the current and expected future value of the portfolio is zero then the expected exposure would be $\sigma\sqrt{t}(T-t)/\sqrt{2\pi}$. Integrating this term between now and the final maturity and dividing by the maturity to find the average EPE would give:

$$\frac{\sigma\int_0^T \sqrt{t}(T-t)}{T\sqrt{2\pi}} = \frac{4}{15\sqrt{2\pi}}\sigma T^{\frac{3}{2}}.$$

ii) Strongly collateralised case

In the collateralised case, a simple proxy for the standard deviation is $\sigma\sqrt{\tau_{MPoR}}(T-t)$ where τ_{MPoR} is the MPoR. Integrating this in a similar manner gives:

$$\frac{\sigma\sqrt{\tau_{MPoR}}\int_0^T (T-t)}{T\sqrt{2\pi}} = \frac{1}{2\sqrt{2\pi}}\sigma T\sqrt{\tau_{MPoR}}$$

iii) Approximate effect of collateral

Taking the ratio of the above EPE terms would give a factor of:

$$\frac{8}{15}\sqrt{T/\tau_{MPoR}} \approx 0.5\sqrt{\frac{T}{\tau_{MPoR}}}.$$

Hence, a useful ballpark estimate of the impact of collateral on reduction of EPE (and CVA) would be by a factor $0.5\sqrt{T/\tau_{MPoR}}$. The ratio is not surprising since the collateral agreement has the impact of reducing the risk horizon from T to τ_{MPoR}. The factor of 8/15 is due to the uncollateralised profile being assumed to have a classic humped shape (obviously for a portfolio with a monotonically increasing exposure such as one dominated by a long-dated cross-currency swap then this factor would be different).

For example, if the margin period of risk was 20 calendar days and the maturity of the portfolio 5-years then the estimate would give 5.09, i.e. the 'collateralised EPE' should be five times smaller than the uncollateralised EPE.

For a cross-currency swap type profile, we can along similar lines compute a multiplier of:

$$\frac{2}{3}\sqrt{\frac{T}{\tau_{MPoR}}}$$

Which is the approximation used in the SA-CCR formula (Section 12.4.2 – footnote 35).

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