APPENDIX 10A: Simple Formula for the EPE and PFE of a Normal distribution

Consider a normal distribution with mean μ (expected future value) and standard deviation (of the future value) σ. The future value of the portfolio in question (for an arbitrary time horizon) is given by:

$$V = \mu + \sigma Z,$$

where Z is a standard normal variable.

i) Potential future exposure (PFE)

This measure is similar to that used in value-at-risk (VAR) calculations. The PFE at a given confidence level α, PFE_α, tells us an exposure that will be exceeded with a probability of no more than $1-\alpha$. For a normal distribution, it is defined by a point a certain number of standard deviations away from the mean:

$$PFE_\alpha = \mu + \sigma\Phi^{-1}(\alpha),$$

where $\Phi^{-1}(.)$ represents the inverse of a cumulative normal distribution function. For example, with a confidence level of $\alpha = 99\%$, we have $\Phi^{-1}(99\%) = 2.33$ and the PFE is 2.33 standard deviations above the expected value (EV) (defined here by μ).

ii) Expected positive exposure (EPE)

The definition of positive exposure is:

$$E = \max(V, 0) = \max\,(\mu + \sigma Z, 0).$$

The EPE (sometimes known as EE) defines the expected value over the positive future values and is therefore:

$$EPE = \int_{-\mu/\sigma}^{\infty} (\mu + \sigma x)\varphi(x)dx = \mu\Phi(\mu/\sigma) + \sigma\varphi(\mu/\sigma),$$

where $\varphi(.)$ represents a normal distribution function and $\Phi(.)$ represents the cumulative normal distribution function. We see that EPE depends on both the mean and the standard deviation; as the standard deviation increases so will the EPE. In the special case of $\mu = 0$, we obtain: $EPE_0 = \sigma\varphi(0) = \sigma/\sqrt{2\pi} \approx 0.4\sigma$. The above formulas are used in Spreadsheet 10.2.

iii) Average EPE

The above analysis is valid only for a single point in time. Suppose we are looking at the whole profile of exposure defined by $V(t) = \sigma\sqrt{t}Z$ where σ represents the annual standard deviation (volatility). The average EPE (AEPE), integrating over time, would be defined by:

$$AEPE \times T = \varphi(0)\int_0^T \sigma\sqrt{t}dt = \frac{1}{\sqrt{2\pi}}\sigma\frac{2}{3}T^{3/2}$$

Leading to:

$$AEPE = \frac{2}{3\sqrt{2\pi}}\sigma\sqrt{T}$$

APPENDIX 10B: Simple Representation of Exposure Profiles

i) Forward

Suppose we want to calculate the exposure on a forward contract and assume the following model for the evolution of the future value of the contract:

$$dV(t) = \mu dt + \sigma dZ,$$

where μ represents a drift and σ is a volatility of the exposure with dZ representing a standard Brownian motion. Under such assumptions the future value at a given time t in the future will follow a normal distribution with known mean and standard deviation:

$$V(t) \sim N\left(\mu t, \sigma\sqrt{t}\right).$$

We therefore have simple expressions for the PFE and EPE following from the formulas in Appendix 10A.

$$PFE_t^{\alpha} = \mu t + \sigma\sqrt{t}\Phi^{-1}(\alpha).$$

$$EPE_t = \mu t \Phi\left(\frac{\mu}{\sigma}\sqrt{t}\right) + \sigma\sqrt{t}\varphi\left(\frac{\mu}{\sigma}\sqrt{t}\right).$$

Spreadsheet 20.2 uses the lognormal version of the above formulas for the analytical PFE.

ii) Swap

Following the above example and assuming zero drift, an approximation to a swap contract is to assume that the future value at a given time is normally distributed according to:

$$V(t) \sim N\left(0, \sigma\sqrt{t}(T - t)\right).$$

where the $(T - t)$ factor corresponds to the approximate duration of the swap of maturity T at time t. This assumes that the expected future value is zero at all future dates which in practice is the case for a flat yield curve (interest rates the same for all maturities). We can show that the maximum exposure is at $s = T / 3$ by differentiating the volatility term:

$$\frac{d}{dt}\left(\sqrt{t_{max}}(T - t_{max})\right) = \frac{1}{2\sqrt{t_{max}}}(T - t_{max}) - \sqrt{t_{max}} = 0.$$

$$t_{max} = T/3.$$

APPENDIX 10C: Exposure Profile for a Cross-Currency Swap

Combined the results in the two previous Appendices, we consider a cross currency swap to be a combination of the approximate FX forward and interest rate swap positions.

The FX forward future value follows $N\left(0,\sigma_{FX}\sqrt{t}\right)$ and each (fixed) interest rate swap follows $N\left(0,\sigma_{IR}\sqrt{t}(T-t)\right)$. Assuming a correlation of ρ between future value of each, the cross-currency swap future value will be given by:

$$V(t) \sim N\left(0, \sqrt{\begin{matrix}\sigma_{FX}^2 t + \sigma_{IR1}^2 t(T-t)^2 + \sigma_{IR2}^2 t(T-t)^2 + 2\rho\sigma_{FX}\sigma_{IR}\ t(T-t) \\ +2\rho\sigma_{FX}\sigma_{IR}\ t(T-t) + 2\rho\sigma_{IR}\ \sigma_{IR}\ t(T-t)^2\end{matrix}}\right).$$

which is used to compute the PFE shown in Spreadsheet 10.4.

APPENDIX 10D: CVA for a CDS contract

Consider a single-name CDS contract with a continuously compounded interest rate r and constant hazard rate of h. For a constant hazard rate (see Appendix 11A), we have the approximation $h = s/LGD$ with s being the running CDS spread and LGD the loss given default.

It is first useful to define the risky duration which is the unit value of the premium leg payments:

$$RD(T) = \int_0^T \exp\left[-(r+h)u\right]du = \frac{1-\exp\left[-(r+h)T\right]}{r+h}$$

The marginal default probability for a given period (Appendix 11A) is:

$$PD(t_{i-1}, t_i) = \exp(-h.t_{i-1}) - \exp(-h.t_i)$$

The EPE for a given period assuming no default is:

$$EPE(t_i) = \frac{s.\sigma_s.\sqrt{t_i}.RD(t_i)}{\sqrt{2\pi}}$$

where σ_s is the assumed spread volatility. A probability weighted EPE is:

$$EPE(t_i) = PD(t_{i-1}, t_i).LGD + [1 - PD(t_{i-1}, t_i)]\frac{s.\sigma_s.\sqrt{t_i}.RD(t_i)}{\sqrt{2\pi}}$$

The PFE at a confidence level of α depends on whether or not the probability of default is greater than the tail probability $(1-\alpha)$:

$$PFE^{\alpha}(t_i) = LGD \qquad 1-\alpha \leq PD(t_{i-1}, t_i)$$

$$PFE^{\alpha}(t_i) = s.\sigma_s.\sqrt{t_i}.RD(t_i).\Phi^{-1}(\alpha) \qquad 1-\alpha > PD(t_{i-1}, t_i)$$

Which is the result of the potential discontinuity in the PFE profile when using a quantile measure. The second formula above would be better represented as $PFE^{\alpha}(t_i) = s.\sigma_s.\sqrt{t_i}.RD(t_i).\Phi^{-1}[\alpha + PD(t_{i-1}, t_i)]$ to account for the default density in the tail of the distribution.

The above is a simplistic treatment that illustrates the general problems with exposure quantification of credit derivatives.

Please also see the following chapter (which is available on www.cvacentral.com) for more details.

https://www.cvacentral.com/wp-content/uploads/2014/05/Counterparty-Risk-in-Credit-Derivative-Contracts-Updated.pdf

Gregory J., 2011, "Counterparty risk in credit derivative contracts", The Oxford Handbook of Credit Derivatives, A. Lipton and A. Rennie (Eds), Oxford University Press.

APPENDIX 10E: Simple Netting Calculation

As discussed in Appendix 10A that the EPE of a normally distributed random variable is:

$$EPE_i = \mu_i \Phi(\mu_i/\sigma_i) + \sigma_i \varphi(\mu_i/\sigma_i).$$

Consider a series of n independent normal variables representing transactions within a netting set (NS). They will have a mean and standard deviation given by:

$$\mu_{NS} = \sum_{i=1}^{n} \mu_i$$

$$\sigma_{NS}^2 = \sum_{i=1}^{n} \sigma_i^2 + 2\sum_{i=1}^{n} \sum_{j=i+1}^{n} \rho_{ij}\sigma_i\sigma_j$$

where ρ_{ij} is the correlation between the (future) values. Assuming normal variables with zero mean and equal standard deviations, $\bar{\sigma}$, and correlations, $\bar{\rho}$, the overall mean and standard deviation are given by:

$$\mu_{NS} = 0 \qquad \sigma_{NS}^2 = (n + n(n-1))\bar{\rho})\bar{\sigma}^2$$

Hence, since $\varphi(0) = 1/\sqrt{2\pi}$, the overall EPE will be:

$$EPE_{NS} = \bar{\sigma}\sqrt{n + n(n-1)\bar{\rho})}/\sqrt{2\pi}.$$

The sum of the individual EPEs gives the result in the case of no netting (NN):

$$EPE_{NS} = \bar{\sigma}n/\sqrt{2\pi}.$$

Hence the netting benefit will be:

$$\frac{EPE_{NS}}{EPE_{NN}} = \sqrt{\frac{1 + (n-1)\bar{\rho})}{n}}$$

For a correlation of $\bar{\rho} = 50\%$ and 10 exposures in the NS, we obtain:

$$\frac{EPE_{NS}}{EPE_{NN}} = \sqrt{\frac{1 + (10-1).0.5)}{10}} = 0.74$$

In the case of perfect positive correlation, $\bar{\rho} = 100\%$, we obtain:

$$\frac{EPE_{NS}}{EPE_{NN}} = \sqrt{\frac{1 + (n-1))}{n}} = 1$$

The maximum negative correlation is bounded by $\bar{\rho} \geq -1/(n-1)$ and we therefore obtain:

$$\frac{EPE_{NS}}{EPE_{NN}} = \sqrt{\frac{1 - \left(n-1\right)/\left(n-1\right)\right)}{n}} = 0$$

APPENDIX 10F: Computation of Marginal Exposure

Suppose we have calculated a netted exposure for a set of trades under a single netting agreement. We would like to be able write the total EPE as a linear combination of EPEs for each trade:

$$EPE_{total} = \sum_{i=1}^{n} EPE_i^*.$$

If there is no netting then we know that the total EPE will indeed be the sum of the individual components and hence the marginal EPE will trivially equal the EPE ($EPE_i^* = EPE_i$). However, since the benefit of netting is to reduce the overall EPE, we expect in the event of netting that $EPE_i^* < EPE_i$. The aim is to find allocations of EPE that reflect a trade's contribution to the overall risk and sum up to the total counterparty level EPE (EPE_{total}).

This type of problem has been studied for other metrics such as value-at-risk (VAR). In the absence of a collateral agreement, EPE (like VAR) is homogenous of degree one which means that scaling the size of the underlying positions by a constant will have the same impact of the EPE. This is written as:

$$\boldsymbol{\alpha} EPE(x) = EPE(\boldsymbol{\alpha} x),$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a vector of weights. By Euler's theorem, we can then define the marginal EPE as:

$$EPE_i^* = \frac{\partial EPE_{total}(\boldsymbol{\alpha})}{\partial \alpha_i}.$$

One way to compute the above partial derivative is to change the size of a transaction by a small value and calculate the marginal EPE using a finite difference. This does not require any additional simulation but just a rescaling of the future values of one trade by an amount $(1 + \varepsilon)$ followed by a recalculation of the EPE for the netting set[1]. The marginal EPE of the trade in question is then given by the change in the EPE divided by ε. The sum of the marginal EPEs will sum to the total EPE[2].

Alternatively, as shown by Rosen and Pykhtin (2010), it can be also computed via a conditional expectation:

$$EPE_i^* = E[\max(V_i, 0)\,|V_{NS} > 0] = S^{-1} \sum_{k=1}^{S} \max\big(V_{i,s}, 0\big)\, I(V_{NS} > 0)$$

where $V_{i,s}$ represents the future value for the transaction i in simulation s (ignoring the time suffix) and $V_{NS} = \sum_{i=1}^{n} V_i$ is the future value for the relevant netting set. The function $I(.)$ is the indicator function that takes the value unity if the statement is true and zero otherwise. Such calculations are illustrated in Spreadsheet 10.6. More detail, including discussion on how to deal with collateralised exposures can be found in Rosen and Pykhtin (2010). The intuition behind the above formula is that the future values of the trade in question are added only if the netting set has positive value at the equivalent point. A trade that has a favourable interaction with the overall netting set may then

[1] ε is a small number such as 0.001.

[2] At least in the current case where no collateral is assumed as discussed below.

have a negative marginal EPE since its future value will be more likely to be negative when the netting set has a positive value.

Whilst marginal EPE is easy to calculate as defined above, it does require the individual trade values in each state to be retained in order to evaluate the formulas above.

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