APPENDIX 17A: CVA formula derivation

We wish to find an expression for the risky value, $V(t,T)$, of a netted set of derivatives positions with a maximum maturity date T. Denote the no default value of the relevant positions as $V_{ND}(t,T)$ and the default time of the counterparty as τ. Then $V(t,T)$ $(t < s \leq T)$ will denote the future value including discounting.

There are two cases to consider:

1) ***Counterparty does not default before*** T

In this case, the risky position is equivalent to the risk-free position and we write the corresponding payoff as:

$$I(\tau > T)V_{ND}(t,T)$$

where $I(\tau > T)$ is the indictor function denoting default (this takes the value 1 if default has not occurred before or at time T and zero otherwise).

1) Counterparty does default before T

In this case, the payoff consists of two terms, the value of the position that would be paid *before* the default time (all cash flows before default will still be paid by the counterparty) plus the payoff at default.

i) Cashflows paid up to the default time

$$I(\tau \leq T)V_{ND}(t,\tau)$$

ii) Default payoff

Here, if the value of the trade at the default time, $V_{ND}(\tau,T)$, is positive then the institution will receive a recovery fraction (R) of the default free value of the derivatives positions whilst if it is negative then they will still have to settle this amount. Hence, the default payoff at time τ is:

$$I(\tau \leq T)[R.V_{ND}(\tau,T)_+ + V_{ND}(\tau,T)_-]$$

where $x_- = \min(x,0)$ and $x_+ = \max(x,0)$.

Putting the above payoffs together, gives the following expression for the actual value including counterparty risk:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau > T)V_{ND}(t,T) + \\ I(\tau \leq T)V_{ND}(t,\tau) + \\ I(\tau \leq T)[R.V_{ND}(\tau,T)_+ + V_{ND}(\tau,T)_-] \end{bmatrix}$$

Re-arranging and using the relationship $x_- = x - x_+$ we obtain:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau > T)V_{ND}(t,T) + \\ I(\tau \leq T)V_{ND}(t,\tau) + \\ I(\tau \leq T)[R.V_{ND}(\tau,T)_+ V_{ND}(\tau,T) - V_{ND}(\tau,T)_+] \end{bmatrix}$$

Since $V_{ND}(t,\tau) + V_{ND}(\tau,T) = V_{ND}(t,T)$ then this becomes:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau > T)V_{ND}(t,T) + \\ I(\tau \leq T)V_{ND}(t,T) + \\ -I(\tau \leq T)[(1-R)V_{ND}(\tau,T)_+] \end{bmatrix}$$

Which simplified to:

$$V(t,T) = V_{ND}(t,T) + E^{Q}\left[-I(\tau \leq T)[(1-R)V_{ND}(\tau,T)_{+}]\right]$$

Note that we made the assumption that the future default free value, $V_{ND}(\tau,T)$, includes discounting for notational simplicity.

 jon@oftraining.com

APPENDIX 17B: CVA as a running spread

Please see the following chapter (which is available on www.cvacentral.com) for more details.

www.cvacentral.com/wp-content/uploads/2014/05/VrinsGregory_RunningCVA.pdf

Vrins, F., and J. Gregory, 2011, "Getting CVA up and running", Risk, October.

APPENDIX 17C: CVA approx. via EPE

Starting from Equation (17.2) for unilateral CVA for a final maturity of T:

$$UCVA(t,T) = -LGD \int_t^T \lambda_C \exp\left(-\int_t^u (r+\lambda_C)ds\right) EPE(t,u)\, du.$$

Assuming the EPE is constant over time (and equal to the average value) leads to:

$$UCVA(t,T) \approx -LGD.EPE. \int_t^T \lambda_C \exp\left(-\int_t^u (r+\lambda_C)ds\right) du.$$

$$EPE = (T-t)^{-1} \int_t^T EPE(t,u)du$$

The term $LGD \int_t^\infty \lambda_C \exp\left(-\int_t^u (r+\lambda_C)ds\right) du$ represents the protection value of a unit CDS contract, $CDS_{protection}(t,T)$. This gives:

$$UCVA(t,T) \approx -EPE.CDS_{protectio}\ (t,T)$$

Since the ratio of CDS protection to the (continuously paid) CDS premium payments (also known as the risky annuity or risky duration) equals the CDS spread (s_{CDS}) then:

$$\frac{UCVA(t,T)}{CDS_{premium}(t,T)} \approx -EPE.\frac{CDS_{protectio}\ (t,T)}{CDS_{premium}(t,T)} \approx -EPE \times s_{CDS}$$

Clearly, the approximation will be a good one if the relationship between EPE, default probability (and indeed discount factors) is reasonably homogeneous through time or other cancellation effects come into play.

See Appendix 12A for more discussion for simple formulas to compute $CDS_{premium}(t,T)$ (the risky annuity).

APPENDIX 17D: Bilateral CVA formula (CVA and DVA)

We wish to find an expression for the risky value, $V(t,T)$, of a netted set of derivatives positions with a maximum maturity date T as in Appendix 17A but under the assumption that the institution concerned may also default in addition to their counterparty. Denoting the default time of the institution as τ_I, their recovery value as R_I and following the notation and logic in Appendix 17A with the counterparty now indexed by C, we now have the following cases:

***1) Neither counterparty nor institution defaults before** T*

In this case, the risky position is equivalent to the default-free position and we write the corresponding payoff as:

$$I(\tau^1 > T)V_{ND}(t,T)$$

where the 'first-to-default' time of the institution and counterparty is defined as $\tau^1 = \min(\tau_C, \tau_I)$.

***2) Counterparty defaults first and also before time** T*

This is the default payoff as in Appendix 17A:

$$I(\tau^1 \leq T)I(\tau^1 = \tau_C)[R_C.V_{ND}(\tau^1,T)_+ + V_{ND}(\tau^1,T)_-]$$

***3) Institution defaults first and also before time** T*

This is an additional term compared with the unilateral CVA case and corresponds to the institution itself defaulting. If they owe money to their counterparty (negative value) then they will pay only a recovery fraction of this whilst if the counterparty owes them money (positive value) then they will still receive this. Hence, the payoff is the opposite of case 2 above:

$$I(\tau^1 \leq T)I(\tau^1 = \tau_I)[R_I.V_{ND}(\tau^1,T)_- + V_{ND}(\tau^1,T)_+]$$

4) If either the institution or counterparty does default then all cashflows prior to the first-to-default date will be paid

$$I(\tau^1 \leq T)V(t,\tau^1)$$

Putting the above payoffs together, we have the following expression for the value including bilateral counterparty risk:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau^1 > T)V_{ND}(t,T) + \\ I(\tau^1 \leq T)V_{ND}(t,\tau^1) + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_C)[R_C.V_{ND}(\tau^1,T)_+ + V_{ND}(\tau^1,T)_-] + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_I)[R_I.V_{ND}(\tau^1,T)_- + V_{ND}(\tau^1,T)_+] \end{bmatrix}$$

Similarly to Appendix 17A, we can simplify the above expression as:

$$V(t,T) = E^Q\begin{bmatrix} I(\tau^1 > T)V_{ND}(t,T) + \\ I(\tau^1 \leq T)V_{ND}(t,\tau^1) + I(\tau^1 \leq T)V_{ND}(\tau^1,T) + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_C)[R_C.V_{ND}(\tau^1,T)_+ - V_{ND}(\tau^1,T)_+] + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_I)[R_I.V_{ND}(\tau^1,T)_- - V_{ND}(\tau^1,T)_-] \end{bmatrix}$$

Finally obtaining:

$$V(t,T) = V_{ND}(t,T) + E^Q\begin{bmatrix} I(\tau^1 \leq T)I(\tau^1 = \tau_C)[R_C.V_{ND}(\tau^1,T)_+ - V_{ND}(\tau^1,T)_+] + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_I)[R_I.V_{ND}(\tau^1,T)_- - V_{ND}(\tau^1,T)_-] \end{bmatrix}$$

 jon@oftraining.com

Under assumptions of no wrong-way risk, this gives the BCVA formula as discussed in Section 17.3.3 including survival probabilities and with risk-free closeout.

 jon@oftraining.com

APPENDIX 17E: Incremental CVA

To calculate incremental CVA for a given netting set (NS), we need to quantify the CVA before (CVA^{NS}) and after (CVA^{NS^*}) a change:

$$CVA^{NS \to NS^*} = -LGD \sum_{i=1}^{m} EPE^{NS^*}(t, t_i) \times PD(t_{i-1}, t_i) - -LGD \sum_{i=1}^{m} EPE^{NS}(t, t_i) \times PD(t_{i-1}, t_i)$$

With the incremental exposure being $EPE^{NS \to NS^*} = EPE^{NS^*} - EPE^{NS}$ then this becomes:

$$CVA^{NS \to NS^*} = -LGD \sum_{i=1}^{m} EPE^{NS \to NS^*}(t, t_i) \times PD(t_{i-1}, t_i)$$

and it is simply necessary to use the incremental EPE in a standard CVA formula.

APPENDIX 17F: Simple wrong-way risk formula

More details of the calculations below can be found in Rosen and Pykhtin (2010).

i) EPE for a forward contract under the assumption of a normally distributed value

In Appendix 11B, we derived a simple formula for the expected exposure (EPE) for an underlying value (V_t) of the form:

$$dV_t = \mu dt + \sigma dW_t,$$

where μ represents a drift and σ is a volatility of the exposure with dW_t representing a standard Brownian motion. The EPE for a given time horizon s is given by:

$$EPE_s = \mu s \Phi\left(\frac{\mu}{\sigma}\sqrt{s}\right) + \sigma s \varphi\left(\frac{\mu}{\sigma}\sqrt{s}\right).$$

ii) EPE expression conditional on default

Now we derive a similar formula but conditioning on some actual default time. Under the above assumptions, the value of a contract at some time s in the future is given by:

$$V(s) = \mu s + \sigma\sqrt{s}Y,$$

where Y is a Gaussian random variable. Let us denote the time of default of the counterparty by τ and the default probability of the counterparty up to time s as $F(s)$ which, as in Appendix 12A, is defined via a constant hazard rate h or intensity of default:

$$F(s) = 1 - \exp(-hs)$$

Like the exposure, default is driven by a Gaussian variable, Z:

$$\tau = F^{-1}(\Phi(Z))$$

Finally, we link the Gaussian variables Y and Z via a correlation parameter ρ:

$$Y = \rho Z + \sqrt{1-\rho^2}\varepsilon,$$

with ε being a further (independent) Gaussian variable. We now need to calculate the EPE conditional upon default having occurred. This is:

$$EPE(s|\tau = s) = E[\max(V(s), 0)|Z = \Phi^{-1}(F(\tau))]$$
$$= \int_{-\mu(t)/\sigma(t)}^{\infty} [\mu'(s) + \sigma'(s)]\varphi(x)dx$$

Where

$$\mu'(s) = \mu(s) - \rho\sigma\Phi^{-1}(F(\tau)) \qquad \sigma'(s) = \sqrt{1-\rho^2}\sigma$$

The conditional EPE is then give by:

$$EPE(s|\tau = s) = \mu'(s)\Phi\left(\frac{\mu'(s)}{\sigma'(s)}\right) + \sigma'(s)\varphi\left(\frac{\mu'(s)}{\sigma'(s)}\right).$$

This is illustrated in Spreadsheet 17.6.

 jon@oftraining.com

APPENDIX 17G: CVA for a CDS contract

Please see the following chapter (which is available on www.cvacentral.com) for more details.

https://www.cvacentral.com/wp-content/uploads/2014/05/Counterparty-Risk-in-Credit-Derivative-Contracts-Updated.pdf

Gregory J., 2011, "Counterparty risk in credit derivative contracts", The Oxford Handbook of Credit Derivatives, A. Lipton and A. Rennie (Eds), Oxford University Press.

