APPENDIX 15A: Swaption analogy and EPE of an interest rate swap

Sorensen and Bollier (1994), effectively show that the CVA of an interest rate swap can be written as:

$$CVA_{swap} = LGD \sum_{i=1}^{n} V_{swaption}(t; t_i, T)\, PD(t_{i-1}, t_i).$$

In the above formula the expected exposure (EPE) is represented by $V_{swaption}(t; t_i, T)$ which is the value today of a European (default free) swaption with maturity t_i on an underlying swap with maturity $T - t_i$. The swaption exercise date of t_i is the potential default time in the discretised CVA formula.

The above shows that the EPE for the purpose of calculating the CVA of the swap can be represented as a series of swaption values. The intuition is that the counterparty has the 'option' to default at any point in the future and therefore effectively cancel the swap.

An interest rate swaption can be priced in a modified Black-Scholes framework via the formula:

$$[F\Phi(d_1) - X\Phi(d_2)]D(t^*, T) \qquad \text{(payer swaption)}$$

$$[-F\Phi(-d_1) + X\Phi(-d_2)]D(t^*, T) \qquad \text{(receiver swaption)}$$

$$d_1 = \frac{\ln\left(\frac{F}{X}\right) + 0.5\sigma_S^2(t^* - t)}{\sigma_S\sqrt{t^* - t}} = d_2 + \sigma_S\sqrt{t^* - t}$$

Where F is the forward rate of the swap, X is the strike (the fixed swap of the underlying swap), σ_S is the swap rate volatility, t^* is the maturity of the swaption (the time horizon of interest). The function $D(t^*, T)$ represents the underlying swap duration (annity value) for which the maturity is $(T - t^*)$. The exposure of the swap will be defined by the interaction between two factors: the swaption payoff, e.g. $F\Phi(d_1) - X\Phi(d_2)$, and the duration $D(t^*, T)$. These quantities respectively increase and decrease monotonically over time. The overall swaption value therefore peaks somewhere in-between. This is illustrated in Spreadsheet 10.1 of the third edition.

In the fourth edition, the swaption example using the Hull and White model is illustrated in Spreadsheet 15.1.

APPENDIX 15B: Marginal EPE

Suppose we have calculated a netted exposure for a set of trades under a single netting agreement. We would like to be able write the total EPE as a linear combination of EPEs for each trade:

$$EPE_{total} = \sum_{i=1}^{n} EPE_i^*.$$

If there is no netting then we know that the total EPE will indeed be the sum of the individual components and hence the marginal EPE will trivially equal the EPE ($EPE_i^* = EPE_i$). However, since the benefit of netting is to reduce the overall EPE, we expect in the event of netting that $EPE_i^* < EPE_i$. The aim is to find allocations of EPE that reflect a trade's contribution to the overall risk and sum up to the total counterparty level EPE (EPE_{total}).

This type of problem has been studied for other metrics such as value-at-risk (VAR). In the absence of a collateral agreement, EPE (like VAR) is homogenous of degree one which means that scaling the size of the underlying positions by a constant will have the same impact of the EPE. This is written as:

$$\boldsymbol{\alpha} EPE(x) = EPE(\boldsymbol{\alpha} x),$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a vector of weights. By Euler's theorem, we can then define the marginal EPE as:

$$EE_i^* = \frac{\partial EPE_{total}(\boldsymbol{\alpha})}{\partial \alpha_i}.$$

One way to compute the above partial derivative is to change the size of a transaction by a small value and calculate the marginal EPE using a finite difference. This does not require any additional simulation but just a rescaling of the future values of one trade by an amount $(1+\varepsilon)$ followed by a recalculation of the EPE for the netting set[1]. The marginal EPE of the trade in question is then given by the change in the EPE divided by ε. The sum of the marginal EPEs will sum to the total EPE[2].

Alternatively, as shown by Rosen and Pykhtin (2010), it can be also computed via a conditional expectation:

$$EPE_i^* = E[\max(V_i, 0)\,|V_{NS} > 0] = S^{-1} \sum_{k=1}^{S} \max\left(V_{i,s}, 0\right) I(V_{NS} > 0)$$

where $V_{i,s}$ represents the future value for the transaction i in simulation s (ignoring the time suffix) and $V_{NS} = \sum_{i=1}^{n} V_i$ is the future value for the relevant netting set. The function $I(.)$ is the indicator function that takes the value unity if the statement is true and zero otherwise. Such calculations are illustrated in Spreadsheet 15.7. More detail, including discussion on how to deal with collateralised exposures can be found in Rosen and Pykhtin (2010). The intuition behind the above formula is that the future values of the trade in question are added only if the netting set has positive value at the equivalent point. A trade that has a favourable interaction with the overall netting set may then

[1] ε is a small number such as 0.001.
[2] At least in the current case where no collateral is assumed as discussed below.

have a negative marginal EPE since its future value will be more likely to be negative when the netting set has a positive value.

Whilst marginal EPE is easy to calculate as defined above, it does require full storage of all the future values at the trade level. From a systems point of view, marginal EPE could be calculated during the overnight batch with little additional effort whereupon the trade-level future values can be discarded. However, for analysing the change in marginal EPE under the influence of a new trade(s) then, unlike incremental EPE, all trade-level values must be retained.

APPENDIX 15C: Collateralised EPE approximation

It is interesting to assess the reduction of EPE due to a collateral agreement as a function of the margin period of risk (MPoR) and maturity of the underlying portfolio. Since CVA is approximately proportional to the EPE then this same reduction can be used to assess the likely impact on CVA. The broad assumptions in deriving this formula are strong collateralisation (zero threshold but no initial margin).

i) Uncollateralised case

As discussed in Appendix 11B, a reasonable proxy for the standard deviation of an uncollateralised portfolio is $\sigma\sqrt{t}(T-t)$ where T is the longest maturity in the portfolio and σ is some volatility term (for example for an interest swap portfolio this would be approximately some weighted average interest rate volatility). Under normal distribution assumptions, assuming the current and expected future value of the portfolio is zero then the expected exposure would be $\sigma\sqrt{t}(T-t)/\sqrt{2\pi}$. Integrating this term between now and the final maturity and dividing by the maturity to find the EPE would give:

$$\frac{\sigma\int_0^T \sqrt{t}(T-t)}{T\sqrt{2\pi}} = \frac{4}{15\sqrt{2\pi}}\sigma T^{\frac{3}{2}}.$$

ii) Collateralised case

In the collateralised case, a reasonable proxy for the standard deviation is $\sigma\sqrt{\tau_{MPoR}}(T-t)$ where τ_{MPoR} is the MPoR. Integrating this in a similar manner gives:

$$\frac{\sigma\sqrt{\tau_{MPoR}}\int_0^T (T-t)}{T\sqrt{2\pi}} = \frac{1}{2\sqrt{2\pi}}\sigma T\sqrt{\tau_{MPoR}}$$

iii) Approximate effect of collateral

Taking the ratio of the above EPE terms would give a factor of:

$$\frac{8}{15}\sqrt{T/\tau_{MPoR}} \approx 0.5\sqrt{\frac{T}{\tau_{MPoR}}}.$$

Hence, a useful ballpark estimate of the impact of collateral on reduction of EPE (and CVA) would be by a factor $0.5\sqrt{T/\tau_{MPoR}}$. The ratio is not surprising since the collateral agreement has the impact of reducing the risk horizon from T to τ_{MPoR}. The factor of 8/15 is due to the uncollateralised profile being assumed to have a classic humped shape (obviously for a portfolio with a monotonically increasing exposure such as one dominated by a long-dated cross-currency swap then this factor should be removed).

For example, if the margin period of risk was 20 calendar days and the maturity of the portfolio 5-years then the estimate would give 5.09, i.e. the 'collateralised EPE' should be five times smaller than the uncollateralised EPE.

For a cross-currency swap type profile, we can along similar lines compute a multiplier of:

$$\frac{2}{3}\sqrt{\frac{T}{\tau_{MPoR}}}$$

APPENDIX 15D: Simple initial margin calculation

As noted in Appendix 11A, the expected positive exposure (EPE) of a normal distribution can be written as:

$$EPE = \mu\Phi\left(\frac{\mu}{\sigma}\right) + \sigma\varphi\left(\frac{\mu}{\sigma}\right)$$

For the collateralised case (zero threshold, no initial margin) the impact of the margin period of risk would lead to $\mu = 0$ and $\sigma = \sqrt{\tau_{MPR}}$ giving an expected exposure (EPE) of:

$$EPE_{no\ IM} = \sqrt{\tau_{MPR}}\varphi(0) = \sqrt{\tau_{MPR}}(2\pi)^{-0.5}$$

The impact of initial margin can be considered equivalent to shifting the mean of the distribution to be $\mu = -\Phi^{-1}(\alpha)\sqrt{\tau_{IM}}$ where τ_{IM} is the time horizon and α the confidence level used to define the initial margin (the initial margin is assumed to be also calculated from normal distribution assumptions potentially using a different time horizon). This leads to an EPE of:

$$EPE_{IM} = -\Phi^{-1}(\alpha)\sqrt{\tau_{IM}}\Phi\left(\frac{-\Phi^{-1}(\alpha)\sqrt{\tau_{IM}}}{\sqrt{\tau_{MPR}}}\right) + \sqrt{\tau_{MPR}}\varphi\left(\frac{-\Phi^{-1}(\alpha)\sqrt{\tau_{IM}}}{\sqrt{\tau_{MPR}}}\right)$$

This can be simplified to give:

$$EPE_{IM} = \sqrt{\tau_{MPR}}\varphi\left(\sqrt{\lambda}K\right) - K\sqrt{\tau_{IM}}\Phi\left(-\sqrt{\lambda}K\right)$$

where $\lambda = \tau_{IM}/\tau_{MPR}$ is the ratio of the time horizon used (τ_{IM}) for the IM calculation divided by the MPR for the exposure quantification (τ_{MPR}) and $K = \Phi^{-1}(\alpha)$ where $\varphi(.)$ is a standard normal density function and $\Phi(.)$ is the cumulative standard normal density function.

Finally:

$$R_{\alpha} = \frac{EPE_{no\ IM}}{EPE_{IM}} = \left[\varphi\left(\sqrt{\lambda}K\right) - K\sqrt{\lambda}\Phi\left(-\sqrt{\lambda}K\right)\right]^{-1}(2\pi)^{-0.5}.$$

 jon@oftraining.com