APPENDIX 12A: Risk-neutral default probability calculation

We refer to the cumulative default probability, $F(t)$, which gives the probability of default any time from now (assuming the counterparty is not currently in default) until time t. This is illustrated in Figure 12.1A. The function must clearly start from zero and tend towards 100% (every counterparty defaults eventually!). A *marginal* default probability, which is then the probability of a default between two specified future dates, is given by:

$$q(t_1, t_2) = F(t_2) - F(t_1) \qquad (t_1 \leq t_2) \tag{12.1}$$

We can see that $F(.)$ must be monotonically increasing to avoid marginal default probabilities being negative.

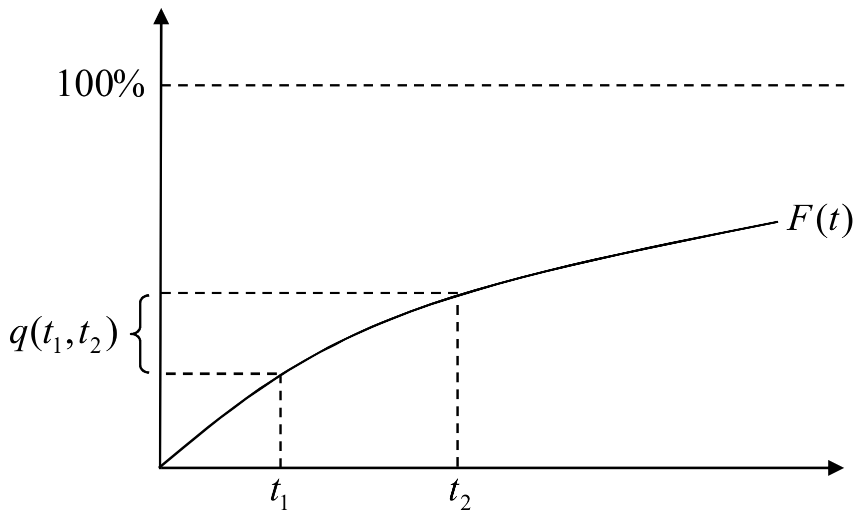

Figure 12.1A. Illustration of cumulative default probability function, $F(t)$, and marginal default probability, $q(t_1, t_2)$.

Suppose a counterparty has a certain constant probability of default each year of (say) 10%. This must be the conditional default probability (i.e., the default probability assuming default has not yet occurred) as otherwise, after more than 10 years, the total default probability would be greater than 100%. This is illustrated in Figure 12.2A. The probability of defaulting in the second year is equal to the probability of surviving the first year and defaulting the next, which would be 90% × 10% = 9%. The probability of defaulting at any time in the first two years is then 10% + 9% = 19%. By similar arguments, the probability of defaulting in the third year must be 90% × 90% × 10% = 8.1%, and so on.

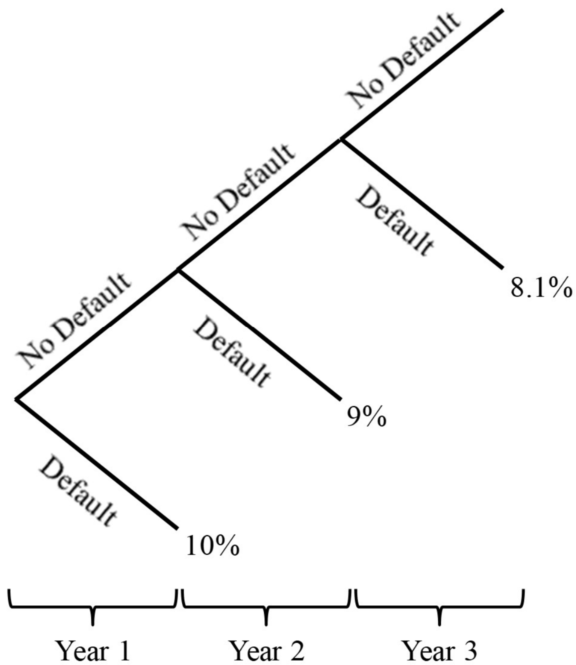

Figure 12.2A. Illustration of the default process through time assuming a conditional default probability of 10% per year.

A more formal mathematical description of the above is that default is driven by a Poisson process and the default probability for a future period u is given by

$$F(u) = 1 - \exp\left(-hu\right), \tag{12.2}$$

where h defines the hazard rate of default, which is the conditional default probability in an infinitesimally small period. By choosing a hazard rate of 10.54%,[1] we can reproduce the results corresponding to the 10% annual default probability; for example, $1 - \exp(-10.54\% \times 2) = 19\%$ is the default probability in the first two years.

An approximate[2] relationship between the hazard rate and credit spread is:

$$h \approx \frac{s}{LGD} \tag{12.3}$$

where the assumed loss given default (LGD) is a percentage. Combining the above two equations gives the following approximate expression for risk-neutral default probability up to a given time u:

$$F(u) = 1 - \exp\left(-\frac{s}{LGD}u\right), \tag{12.4}$$

Combination of equations (12.1) and (12.4) leads to equation (12.1) given in the book.

The reason that risk-neutral default probability depends on LGD can be explained as follows. Suppose a bond will default with a probability of 2% but the LGD would be 50%. The expected loss is 1%, which is the same as if the bond had a 1% probability of default but the LGD was 100%. In the market we see only a single parameter (the

[1] This can be found from inverting equation (10.2) at the 1-year point as –log(1-10%) where log represents the natural logarithm.

[2] This assumes that the credit spread term structure is flat (credit spreads for all maturities are equal) and that CDS premiums are paid continuously.

credit spread) and must imply two values from it. Common practice is then to fix the LGD and derive the default probability. A lower LGD must be balanced (good for the bondholder) by a larger assumed default probability (bad for the bondholder).

The above formula is a good approximation generally, although to compute the implied default probabilities accurately we must solve numerically for the correct hazard rate, assuming a certain underlying functional form. The reader is referred to O'Kane (2008) for a more detailed discussion. Such an approach is also required to take into account the term structure of credit spreads and incorporate other aspects such as the convention of using upfront premiums in the CDS market.

It is sometimes useful to define the risky annuity in order to make simple calculations (for example, Section 17.2.3 in the book). This is the value of receiving a unit at periodic intervals in the future as long as a counterparty does not default. A simple approximation for this expression using a constant interest rate and hazard rate and assuming continuous premium payments is:

$$\begin{aligned}\int_0^T \exp(-ru)\exp(-hu)\,du &= \int_0^T \exp(-(r+h)u)\,du \\ &= \frac{1-\exp\left(-(r+h)T\right)}{r+h}\end{aligned} \tag{12.5}$$

In a similar way, the value of protection in the CDS contract, assuming default can occur at any point in time and results in a payment of an amount equal to the LGD is:

$$LGD\int_0^T \exp(-ru)\,h\exp(-hu)\,du = \frac{h\left[1-\exp\left(-(r+h)T\right)\right]}{r+h}LGD \tag{12.6}$$

The ratio of Equation (12.6) to (12.5) should be equal to the CDS premium which is another way to see the relationship $h \approx \frac{s}{LGD}$.

To allow for a term structure of credit (for example, CDS premiums at different maturities) and indeed a term structure of interest rates, we must choose some functional form for h. Such an approach is the credit equivalent of yield curve stripping and was first suggested by Li (1998). The single-name CDS market is mainly based around 5-year instruments and other maturities will be rather illiquid. A standard approach is to choose a piecewise constant representation of the hazard rate to coincide with the maturity dates of the individual CDS quotes. This is illustrated in Spreadsheet 12.1.