APPENDIX 12A: Deriving the standard CVA formula.

We wish to find an expression for the risky value, $\tilde{V}(t,T)$, of a netted set of derivatives positions with a maximum maturity date T. Denote the risk-free value (current MtM) of the relevant positions as $V(t,T)$ and the default time of the counterparty as τ. Then $V(s,T)$ $(t < s \leq T)$ will denote the future uncertain MtM accounting for discounting effects.

There are two cases to consider:

1) *Counterparty does not default before T*

In this case, the risky position is equivalent to the risk-free position and we write the corresponding payoff as:

$$I(\tau > T)V(t,T),$$

Where $I(\tau > T)$ is the indictor function denoting default (this takes the value 1 if default has not occurred before or at time T and zero otherwise).

2) *Counterparty does default before T*

In this case, the payoff consists of two terms, the value of the position that would be paid *before* the default time (all cash flows before default will still be paid by the counterparty) plus the payoff at default.

i) Cashflows paid up to the default time

$$I(\tau \leq T)V(t,\tau)$$

ii) Default payoff

Here, if the MtM of the trade at the default time, $V(\tau,T)$, is positive then the institution will receive a recovery fraction (R) of the risk-free value of the derivatives positions whilst if it is negative then they will still have to settle this amount. Hence, the default payoff at time τ is:

$$I(\tau \leq T)\left(RV(\tau,T)^{+} + V(\tau,T)^{-}\right),$$

where $x^{-} = \min(x,0)$ and $y^{+} = \max(y,0)$.

Putting the above payoffs together, we have the following expression for the value of the risky position under the risk-neutral measure:

$$\tilde{V}(t,T) = E^Q\begin{bmatrix} I(\tau > T)V(t,T) + \\ I(\tau \le T)V(t,\tau) + \\ I(\tau \le T)\left(RV(\tau,T)^+ + V(\tau,T)^-\right) \end{bmatrix}.$$

The above expression is general but not especially useful or even insightful. However, re-arranging and using the relationship $x^- = x - x^+$ we obtain:

$$\begin{aligned}
&\tilde{V}(t,T) \\
&= E^Q\begin{bmatrix} I(\tau > T)V(t,T) + \\ I(\tau \le T)V(t,\tau) + \\ I(\tau \le T)\left(RV(\tau,T)^+ + V(\tau,T) - V(\tau,T)^+\right) \end{bmatrix}. \\
&= E^Q\begin{bmatrix} I(\tau > T)V(t,T) + \\ I(\tau \le T)V(t,\tau) + \\ I(\tau \le T)\left((R-1)V(\tau,T)^+ + V(\tau,T)\right) \end{bmatrix}
\end{aligned}$$

Now, realising that we can combine two terms since $V(t,\tau) + V(\tau,T) \equiv V(t,T)$ we have:

$$\tilde{V}(t,T) = E^Q\begin{bmatrix} I(\tau > T)V(t,T) + \\ I(\tau \le T)V(t,T) + \\ I(\tau \le T)\left((R-1)V(\tau,T)^+\right) \end{bmatrix}.$$

Finally, since $I(\tau > T)V(t,T) + I(\tau \le T)V(t,T) \equiv V(t,T)$, we have:

$$\tilde{V}(t,T) = V(t,T) - E^Q\left[(1-R)I(\tau \le T)V(\tau,T)^+\right].$$

The above equation is crucial since it defines the risky value of a netting set of derivatives positions with respect to the risk-free value. The relevant term is often known as CVA (credit value adjustment). It is an adjustment to the risk-free value of the positions within the netting set to account for counterparty risk:

$$\tilde{V}(t,T) = V(t,T) - CVA(t,T).$$

$$CVA(t,T) = E^Q\left[(1-R)I(\tau \le T)V(\tau,T)^+\right].$$

Note that we made the assumption that the future MtM value, $V(s,T)$, includes discounting for notational simplicity. If we drop this assumption, the above formula will include discounting:

$$CVA(t,T) = E^Q\left[(1-R)I(\tau \le T)V(\tau,T)^+ \frac{\beta(t)}{\beta(\tau)}\right],$$

where $\beta(s)$ is the value of the "money market account" at time s.

APPENDIX 12B: Computation of the CVA formula and simple spread-based approximation.

To derive the classic CVA formula we can then write:

$$CVA(t,T) = (1-\overline{R})E^{Q}\left[I(u \le T)V^{*}(u,T)^{+}\right],$$

Where $\overline{R}$ is the mean or expected recovery value. We use $V^{*}(u,T)$ to denote:

$$V^{*}(u,T) = V(u,T) \mid \tau = u \,.$$

This is a critical point in the analysis as the above statement requires the exposure at a future date, $V(u,T)$, *knowing* that default of the counterparty has occurred at that date $(\tau = u)$. Ignoring wrong-way risk simple sets $V^{*}(u,T) = V(u,T)$ which we do from now on.

Since the expectation in the above equation is over all times before the final maturity, we can integrate over all possible default times. We obtain:

$$CVA(t,T) = -(1-\overline{R})E^{Q}\left[\int_{t}^{T} B(t,u)V(u,T)^{+}\,dF(t,u)\right],$$

where $B(t,u)$ is the risk-free discount factor and $F(t,u)$ is the cumulative default probability for the counterparty (probability of no default) as described in Appendix 10B.

We recognise the discounted expected exposure calculated under the risk-neutral measure denoted by $EE_d(u,T) = E^{Q}\left[B(t,u)V(u,T)^{+}\right]$. Assuming that the default probabilities are deterministic, we have:

$$CVA(t,T) = (1-\overline{R})\left[\int_{t}^{T} EE_d(u,T)dF(t,u)\right],$$

Finally, we could compute the above equation via some integration scheme such as:

$$CVA(t,T) \approx (1-\overline{R})\sum_{i=1}^{m} EE_d(t,t_i)\left[F(t,t_i) - F(t,t_{i-1})\right],$$

where we have m periods given by $[t_0(=t), t_1, ..., t_m(=T)]$. As long as m is reasonably large then this will be a good approximation.

With further simplifying assumptions, one can obtain a simple expression for CVA linked to the credit spread of the counterparty. To do this we have to work with the non-discounted expected exposure[1] as therefore start from the following CVA formula:

$$CVA(t,T) = (1-\overline{R})\left[\int_t^T EE(u,T)B(t,u)dF(t,u)\right]$$

Suppose that we approximate the (undiscounted) expected exposure term, $EE(u,T)$, as a fixed known amount which would obviously be the EPE. The fixed EPE (defined in Chapter 8) would most obviously be computed from the EE averaged over time, for example:

$$EPE = \frac{\int_t^T EE(u,T)du}{T-t} \approx \frac{1}{m}\sum_{j=1}^{m} EE(t,t_j)$$

Clearly, the approximation will be a good one if the relationship between EPE, default probability (and indeed discount factors) is reasonably homogeneous through time or other cancellation effects come into play. Using this approach, the CVA is:

$$CVA(t,T) \approx (1-\overline{R})E^Q\left[\int_t^T B(t,u)dF(t,u)\right]EPE$$

.

Recalling Appendix 10B, we can see this is simply the value of CDS protection on a notional equally the EPE. Hence we have the following approximation giving a running CVA (i.e. expressed as a spread):

$$CVA \approx EPE \times Spread\ .$$

We can use the risky annuity formulas (Appendix 10B) to convert this to an up-front value.

[1] As described in Chapter 12, this can be done if EE is computed using the T-forward measure.

APPENDIX 12C: CVA formula for an option position.

In this case we have a simplification since the exposure of the long option position can never be negative:

$$CVA_{option}(t,T) = (1-\overline{R})E^{Q}\left[I(\tau \leq T\right]E^{Q}\left[B(t,\tau)V_{option}(\tau,T)\right]$$
$$= (1-\overline{R})F(t,T)V_{option}(t,T),$$

where $V_{option}(t,T)$ is the upfront premium for the option. This means that the value of the risky option can be calculated as:

$$V_{option}(t,T) - CVA_{option}(t,T) = V_{option}(t,T) - (1-\overline{R})F(t,T)V_{option}(t,T)$$
$$= V_{option}(t,T)\left[1-F(t,T)\right] + \overline{R}V_{option}(t,T)F(t,T).$$

With zero recovery we have simply that the risky premium is the risk-free value multiplied by the survival probability over the life of the option.

APPENDIX 12D: Semi-analytical calculation of the CVA for a swap.

As noted by Sorensen and Bollier (1994), the CVA of a swap position can be written as:

$$CVA_{swap} \approx (1-\overline{R})\sum_{j=1}^{m}\left[F(t,t_j)-F(t,t_{j-1})\right]V_{swaption}(t;t_j,T),$$

where $V_{swaption}(t;s,T)$ is the value today of the reverse swap with maturity date T and exercise date t_j. The intuition is that the counterparty has the "option" to default at any point in the future and therefore cancel the trade (execute the reverse position). The values of these swaptions are weighted by the relevant default probabilities and recovery is taken into account. Not only is this formula useful for analytical calculations, it is also quite intuitive for explaining CVA.

An interest rate swaption can be priced in a modified Black-Scholes framework via the formula:

$$\left(F\Phi(d_1)-X\Phi(d_2)\right)D(t^*,T) \qquad \text{(payer swaption)}$$
$$\left(-F\Phi(-d_1)+X\Phi(-d_2)\right)D(t^*,T) \qquad \text{(receiver swaption)}$$

$$d_1 = \frac{\log(F/X)+0.5\sigma_S{}^2(t^*-t)}{\sigma\sqrt{t^*-t}} = d_2 + \sigma_S\sqrt{t^*-t}\,.$$

Where F is the forward rate of the swap, X is the strike rate (the fixed swap of the underlying swap), σ_S is the swap rate volatility, t^* is the maturity of the swaption (the time horizon of interest), $T-t^*$ will be the maturity of the underlying swap. The exposure of the swap will be defined by the interaction between two factors: the swaption payoff, $F\Phi(d_1)-X\Phi(d_2)$, and the underlying swap duration (annity), $D(t^*,T)$. These quantities respectively increase and decrease monotonically over time. The overall swaption value therefore peaks somewhere in-between as illustrated in Figure 12.6 and Spreadsheet 12.2.

APPENDIX 12E: Incremental CVA formula.

To calculate incremental CVA, we need to quantify the change before and after added a new trade, *i*:

$$
\begin{aligned}
&CVA^{NS+i}(t,T) - CVA^{NS}(t,T) \\
&= (1-\overline{R})\sum_{i=1}^{m} EE^{NS+i}(t,t_i)\left[F(t,t_i) - F(t,t_{i-1})\right] - (1-\overline{R})\sum_{i=1}^{m} EE^{NS}(t,t_i)\left[F(t,t_i) - F(t,t_{i-1})\right] \\
&= (1-\overline{R})\sum_{i=1}^{m} \left[EE^{NS+i}(t,t_i) - EE^{NS}(t,t_i)\right]\left[F(t,t_i) - F(t,t_{i-1})\right]
\end{aligned}
$$

We therefore simply need to use the incremental EE, $EE^{NS+i}(t,t_i) - EE^{NS}(t,t_i)$ in the standard CVA formula.