APPENDIX 9A: Additional mathematical detail on exposure models.

i) Interest rates

In the Hull and White model, the "short rate" (short-term interest rate) is assumed to follow the following process:

$$dr_t = [\theta(t) - ar_t]dt + \sigma_r dW_t,$$

which is a Brownian motion with mean reversion. Mean reversion dictates that when the rate is above some "mean" level, it is pulled back towards that level with a certain force according to the size of parameter a. The mean reversion level $\theta(t)$ is time-dependent which is what allows this model to be fitted to the initial yield curve. The mean reversion has the effect of damping the standard deviation of discount factors, $B(t,T)$, written as:

$$\sigma(B(t,T)) = \sigma_r \left[\frac{1 - \exp(-a(T-t))}{a(T-t)} \right].$$

Although the yield curve is not modelling directly, it can be "reconstructed" at any point, given knowledge of the above parameters and the current short rate. Hence, using such an approach in a Monte Carlo simulation is relatively straightforward. Using historical data, one can estimate the standard deviation (volatility) of zero-coupon bond prices of various maturities, then it is possible to estimate values for σ_r and a. For risk-neutral pricing, these parameters would be derived from the prices of instrumental such as interest-rate swaptions. Spreadsheet 9.1 uses this model for a flat yield curve (Vasicek model).

ii) Equity

A standard geometric Brownian motion (GBM) is defined by:

$$\frac{dS_t}{S_t} = \mu(t)dt + \sigma_E(t)dW_t,$$

where S_t represents the value of the equity in question at time t, $\mu(t)$ is the drift, $\sigma_E(t)$ is the volatility and dW_t is a standard Brownian motion.

iii) FX

FX may be modelled via a GBM but for long time horizons then mean reversion may be relevant. The equation can then be written:

$$\frac{dX_t}{X_t} = k(\theta - \ln X_t)dt + \sigma_{FX}(t)dW_t$$

Where k is the rate of mean reversion to a long-term mean level θ.

iv) Commodities

A simple and popular model (see Geman 2005) is:

$$\ln S_t = f(t) + Z(t)$$
$$dZ(t) = \left(\alpha - \beta Z(t)\right)dt + \sigma_C(t)dW_t$$

Where $f(t)$ is a deterministic function, which may be expressed using sin or cos trigonometry functions to give the relevant periodicity and the parameters α and β are the mean reversion parameters.

v) Credit

A typical model including jumps could be:

$$d\lambda_t = \theta(\eta - \lambda_t)dt + \sigma_\lambda \sqrt{\lambda_t} dW_t + jdN\ ,$$

where λ_t is the intensity (or hazard rate) of default[1] and θ and η are mean reversion parameters. Additionally, dN represents a Poisson jump with jump size j. This jump size can itself be random such as following an exponential distribution.

[1] This means that the default probability in a period dt conditional on no default before time t is $\lambda_t dt$. The intensity of default is closely related to the credit spread as is explained in Chapter 10.

APPENDIX 9B: Marginal exposure calculation.

Suppose we have calculated a netted exposure for a set of trades under a single netting agreement. We would like to be able write the total EE as a linear combination of EEs for each trade, i.e.:

$$EE_{total} = \sum_{i=1}^{n} EE_i^* .$$

As described in Chapter 12, marginal CVA calculations follow trivially. If there is no netting then we know that the total EE will indeed be the sum of the individual components and hence the marginal EE will equal the EE. However, since the benefit of netting is to reduce the overall EE, we expect in the event of netting that $EE_i^* < EE_i$. In the case of perfectly offsetting exposures, the marginal EEs must sum to zero.

The aim is to find allocations of EE that reflect a trade's contribution to the overall risk and sum up to the counterparty level EE, (EE_{total}).

$$EE_{total} = \sum_{i=1}^{n} EE_i^* .$$

In the absence of a collateral agreement, expected exposure (like VAR) is homogenous of degree one which means

$$\boldsymbol{\alpha} EE(x) = EE(\boldsymbol{\alpha} x) ,$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a weight vector. By Euler's theorem we can then define the marginal EE as: -

$$EE_i^* = \frac{\partial EE_{total}(\boldsymbol{\alpha})}{\partial \alpha_i} .$$

As shown by Rosen and Pykhtin (2010), this can be also computed via a conditional expectation:

$$EE_i^* = E\left[\max(V_i, 0) \mid V_{NS} > 0\right],$$

where $V_{NS} = \sum_{i=1}^{n} V_i$ is the future value for the relevant netting set. Marginal exposure calculations are illustrated in Spreadsheet 9.5. More detail, including discussion on how to deal with collateralised exposures can be found in Rosen and Pykhtin (2010).

APPENDIX 9C: Example calculations of the impact of collateral on exposure.

Assuming that a netted set of trades is perfectly collateralised at a given time and the change in the netted exposure (and collateral value) follows a normal distribution with zero mean and volatility parameter σ_E, then using the results of Appendix 8A, the potential future exposure at as given confidence level α is given by:

$$PFE_\alpha = \Phi^{-1}(\alpha) \times \sigma_E \times \sqrt{\Delta t} ,$$

where Δt denotes the margin period of risk. The above formula is analogous to a VAR formula under a normal distribution assumption of portfolio value. The EE is given by:

$$EE = \sigma_E \sqrt{\Delta t} \varphi(0) = \frac{1}{\sqrt{2\pi}} \times \sigma_E \times \sqrt{\Delta t} \approx 0.4 \sigma_E \sqrt{\Delta t} .$$

Given the short period, it is unlikely that the drift of the distribution is likely to be an important consideration.

In the case of a swap, the duration, must be considered in order to derive a price volatility. A simple way to do this is to multiply by the remaining maturity giving the approximation shown in Equation (9.5) of Chapter 9:

$$EE(u) = 0.4 \times \sigma_E \times \sqrt{\Delta t} (T - u) .$$

This approximation can be extended to include the relevant forward rates although it clearly ignores the volatility of exposure that a Monte Carlo approach would capture.