APPENDIX 13A: Deriving the bilateral CVA formula.

We wish to find an expression for the risky value, $\tilde{V}(t,T)$, of a netted set of derivatives positions with a maximum maturity date T as in Appendix 12A but under the assumption that the institution concerned may also default in addition to their counterparty. Denoting the default time of the institution as τ_I, their recovery value as R_I and following the notation and logic in Appendix 12A, we now have the following cases (we denote the "first-to-default time" of the institution and counterparty as $\tau^1 = \min(\tau, \tau_I)$).

1) Neither counterparty nor institution defaults before T

In this case, the risky position is equivalent to the risk-free position and we write the corresponding payoff as:

$$I(\tau^1 > T)V(t,T).$$

2) Counterparty defaults first and also before time T.

This is the default payoff as in Appendix 7.A:

$$I(\tau^1 \leq T)I(\tau^1 = \tau)\left(RV(\tau^1,T)^+ + V(\tau^1,T)^-\right).$$

3) Institution defaults first and also before time T

This is an additional term compared with the unilateral CVA case and corresponds to the institution itself defaulting. If they owe money to their counterparty (negative MtM) then they will pay only a recovery fraction of this whilst if the counterparty owes them money (positive MtM) then they will still receive this. Hence, the payoff is the opposite of case 2 above:

$$I(\tau^1 \leq T)I(\tau^1 = \tau_I)\left(R_I V(\tau_I,T)^- + V(\tau_I,T)^+\right).$$

4) If either the institution or counterparty does default then all cashflows prior to the first-to-default date will be paid

$$I(\tau^1 \leq T)V(t,\tau^1).$$

Putting the above payoffs together, we have the following expression for the value of the risky position:

$$\tilde{V}(t,T) = E^Q\begin{bmatrix} I(\tau^1 > T)V(t,T) + \\ I(\tau^1 \leq T)V(t,\tau^1) + \\ I(\tau^1 \leq T)I(\tau^1 = \tau)\left(RV(\tau^1,T)^+ + V(\tau^1,T)^-\right) + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_I)\left(R_I V(\tau^1,T)^- + V(\tau^1,T)^+\right) \end{bmatrix}.$$

Similarly to Appendix 7.A, we simplify the above expression as:

$$\tilde{V}(t,T) = E^Q \begin{bmatrix} I(\tau^1 \le T)V(t,T) + \\ I(\tau^1 \le T)V(t,\tau^1) + I(\tau^1 = \tau)V(\tau^1,T) + I(\tau^1 = \tau_I)V(\tau^1,T) + \\ I(\tau^1 \le T)I(\tau^1 = \tau)\left(RV(\tau^1,T)^+ - V(\tau^1,T)^+\right) + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)\left(R_I V(\tau^1,T)^- - V(\tau^1,T)^-\right) \end{bmatrix}$$

Finally obtaining:

$$\tilde{V}(t,T) = V(t,T) + E^Q \begin{bmatrix} I(\tau^1 \le T)I(\tau^1 = \tau)\left(RV(\tau^1,T)^+ - V(\tau^1,T)^+\right) + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)\left(R_I V(\tau^1,T)^- - V(\tau^1,T)^-\right) \end{bmatrix}.$$

$$\tilde{V}(t,T) = V(t,T) - E^Q \begin{bmatrix} I(\tau^1 \le T)I(\tau^1 = \tau)(1-R)V(\tau^1,T)^+ + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)(1-R_I)V(\tau^1,T)^- \end{bmatrix}.$$

We can identify the BCVA (bilateral CVA) term as being:

$$BCVA(t,T) = E^Q \begin{bmatrix} I(\tau^1 \le T)I(\tau^1 = \tau)(1-R)V(\tau^1,T)^+ + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)(1-R_I)V(\tau^1,T)^- \end{bmatrix}.$$

Finally, under the similar assumptions of no wrong-way risk and of no simultaneous default between the default of the institution and its counterparty, we would have a formula analogous to that derived in Appendix 12B for computing BCVA:

$$\begin{aligned} & BCVA(t,T) = \\ & -(1-\bar{R})E^Q\left[\int_t^T B(t,u)V(u,T)^+ S_I(u)dS(t,u)\right] \\ & +(1-R_I)E^Q\left[\int_t^T B(t,u)V(u,T)^- S(u)dS_I(t,u)\right] \end{aligned}.$$

An obvious approximation to compute this formula using the discounting EE and NEE would then be:

$$\begin{aligned} & BCVA(t,T) \approx (1-\bar{R})\sum_{i=1}^{m} EE_d(t,t_i)S_I(t,t_{i-1})\left[F(t,t_i) - F(t,t_{i-1})\right] \\ & -(1-\bar{R}_I)\sum_{i=1}^{m} NEE_d(t,t_i)S(t,t_{i-1})\left[F_I(t,t_i) - F_I(t,t_{i-1})\right] \end{aligned}.$$

More details on these calculations and discussion on incorporating dependency between the default of the institution and the counterparty can be found in Gregory (2009a).

A simple spread based approximation would be:

$$CVA \approx EPE \times Spread - ENE \times Spread_I \, ,$$

where $Spread_I$ represents the credit spread of the institution themselves.