APPENDIX 11A: Calculation of joint default probabilities with a bivariate normal distribution function and description of credit portfolio models.

i) Two-counterparty case

The modelling of joint default events is conceptually difficult due to the binary[1] nature of default and the lack of data regarding default events, let alone joint default events. What is required is an intuitive way to generate dependence between defaults with some underlying economic structure. This has been classically achieved using a Merton inspired approach. In the Merton model we simply write default as being the point at which a process for the asset value falls below a certain "default threshold" as illustrated in Figure 11.2 in the book.

One can interpret X as being an asset value in the classic Merton sense with the change in the asset value during a small time interval, the asset return, assumed to follow a standard normal distribution. Assuming the default probability of the firm is already known then the precise distributional specification of the asset return (such as the drift and volatility) is unimportant. One must set the default barrier in order to retrieve the correct default probability which corresponds to $k = \Phi^{-1}(p)$, where $\Phi^{-1}(.)$ is the inverse of a cumulative normal distribution function and p is the default probability of the name over the time horizon of interest.

The above modelling framework[2] might first appear to be nothing more than a mapping exercise. However, the power of the approach is the elegance and intuition when introducing another default event. Now the joint default probability can be defined by a two-dimensional or bivariate Gaussian distribution function. Hence, in the "double default" model of interest, the joint default probability of two names A and X, p_{AX}, is given by: -

$$p_{AX} = \Phi_2(k_A, k_X; \rho_{AX}),$$

Where Φ_2 is a cumulative bivariate cumulative distribution function, ρ_{AX} is the correlation between A and B, often referred to as the "asset correlation" and k_A and k_X are the default barriers as defined above. An illustration of this is given in 11.3 in the book.

ii) Many counterparties

A Monte Carlo implementation of the above is rather straightforward (as can be seen in Spreadsheet 11.2) and consists of the following steps:

[1] Meaning simply that default can either occur or not, so there are just two (rather than a continuum of) states to consider.

[2] Although there is a clear link between this simple approach and the Merton model, we have ignored the full path of the asset value process and linked default to just a single variable. A more rigorous approach, however, does not differ significantly and is much more complex to implement (see Hull et al., 2004).

1. Simulation of a vector of N-correlated Gaussian variables denoted by $\left(X_1,......,X_N\right)$. This can be done using methods such as the Box-Muller transform and Cholesky decomposition (see Press et al., 2007).
2. For each variable
    a. Check if $X_i < k_i$ in which case the name has defaulted
    b. If the name has defaulted then draw a possible value from the exposure distribution.
    c. If the above value is positive then multiply by the loss given default and update the total loss accordingly.

3. Proceed to the next simulation.

Monte Carlo simulation is completely general and easy to implement but is extremely slow. Monte Carlo acceleration techniques have been developed (for example, see Glasserman and Li, 2005). In addition, faster (but approximate) analytical approaches have been described for computing losses. We will review these approaches briefly next.