APPENDIX 21A: Beta hedging

i) Single asset

Suppose we hedge a unit amount of an asset A with an amount β of another asset B. The volatilities of the assets are σ_A and σ_B and their correlation is ρ_{AB}. The total variance will be:

$$\sigma_A^2 + \beta^2\sigma_B^2 + 2\rho_{AB}\beta\sigma_A\sigma_B.$$

The optimal amount β that minimises the total variance is determined by differentiation:

$$2\beta\sigma_B^2 + 2\rho_{AB}\sigma_A\sigma_B = 0.$$

Which gives the solution

$$\beta = -\rho_{AB}\frac{\sigma_A}{\sigma_B}.$$

This, in turn, leads to a variance of:

$$\sigma_A^2 + \rho_{AB}^2\frac{\sigma_A^2}{\sigma_B^2}\sigma_B^2 - 2\rho_{AB}\rho_{AB}\frac{\sigma_A}{\sigma_B}\sigma_A\sigma_B = \sigma_A^2 - \rho_{AB}^2\sigma_A^2 = \sigma_A^2(1-\rho_{AB}^2).$$

The optimal hedge therefore leads to a reduction in variance of $(1-\rho_{AB}^2)$ or reduction in standard deviation of $\sqrt{1-\rho_{AB}^2}$.

ii) Homogeneous portfolio

Now suppose there is a homogenous portfolio of n assets each with volatility σ_A being hedged with another single asset B. The return on asset B will be a Gaussian variable V_B whilst the return of each of the assets in the portfolio is:

$$V_i = \rho V_B + \sqrt{1-\rho^2}V_i'$$

where the V_i' are uncorrelated Gaussian variables. The correlation of each asset in the portfolio to the asset B is ρ whilst the correlation between different assets is ρ^2. The total change in portfolio value is:

$$\sum_i \sigma_A V_i + \beta\sigma_B V_B = \sum_i \sigma_A\left(\rho V_B + \sqrt{1-\rho^2}V_i'\right) + \beta\sigma_B V_B$$
$$= (n\sigma_A\rho + \beta\sigma_B)V_B + \sum_i \sigma_A\sqrt{1-\rho^2}V_i'$$

The variance is:

$$(n\sigma_A\rho + \beta\sigma_B)^2 + n(1-\rho^2)\sigma_A^2$$

The first term above can be seen as systematic risk arising from the relationship to the index and the second is idiosyncratic risk.

With no hedge ($\beta = 0$) then the variance is:

$$(n\sigma_A\rho)^2 + n(1-\rho^2)\sigma_A = \sigma_A^2(n^2\rho^2 + n - n\rho^2)$$

With the variance minimising hedge ($\beta = -n\rho\sigma_A/\sigma_B$) then the variance is:

$$n(1-\rho^2)\sigma_A^2$$

The ratio (hedging benefit) as shown in Figure 21.5 is therefore the square root of:

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$$\frac{(1-\rho^2)}{(1-\rho^2)+n\rho^2}$$