APPENDIX 14A: Deriving the standard CVA formula.

We wish to find an expression for the risky value, $\tilde{V}(t,T)$, of a netted set of derivatives positions with a maximum maturity date T. Denote the risk-free value (current MtM) of the relevant positions as $V(t,T)$ and the default time of the counterparty as τ. Then $V(s,T)$ $(t < s \leq T)$ will denote the future uncertain MtM accounting for discounting effects.

There are two cases to consider:

1) Counterparty does not default before T

In this case, the risky position is equivalent to the risk-free position and we write the corresponding payoff as:

$$I(\tau > T)V(t,T),$$

Where $I(\tau > T)$ is the indictor function denoting default (this takes the value 1 if default has not occurred before or at time T and zero otherwise).

2) Counterparty does default before T

In this case, the payoff consists of two terms, the value of the position that would be paid <u>*before*</u> the default time (all cash flows before default will still be paid by the counterparty) plus the payoff at default.

i) Cashflows paid up to the default time

$$I(\tau \leq T)V(t,\tau)$$

ii) Default payoff

Here, if the MtM of the trade at the default time, $V(\tau,T)$, is positive then the institution will receive a recovery fraction (R) of the risk-free value of the derivatives positions whilst if it is negative then they will still have to settle this amount. Hence, the default payoff at time τ is:

$$I(\tau \leq T)\left(RV(\tau,T)^{+} + V(\tau,T)^{-}\right),$$

where $x^{-} = \min(x,0)$ and $y^{+} = \max(y,0)$.

Putting the above payoffs together, we have the following expression for the value of the risky position under the risk-neutral measure:

$$\tilde{V}(t,T) = E^{Q}\begin{bmatrix} I(\tau > T)V(t,T) + \\ I(\tau \leq T)V(t,\tau) + \\ I(\tau \leq T)\left(RV(\tau,T)^{+} + V(\tau,T)^{-}\right) \end{bmatrix}.$$

The above expression is general but not especially useful or even insightful. However, re-arranging and using the relationship $x^- = x - x^+$ we obtain:

$$\begin{aligned}
&\tilde{V}(t,T) \\
&= E^Q\begin{bmatrix} I(\tau > T)V(t,T) + \\ I(\tau \le T)V(t,\tau) + \\ I(\tau \le T)\big(RV(\tau,T)^+ + V(\tau,T) - V(\tau,T)^+\big) \end{bmatrix}. \\
&= E^Q\begin{bmatrix} I(\tau > T)V(t,T) + \\ I(\tau \le T)V(t,\tau) + \\ I(\tau \le T)\big((R-1)V(\tau,T)^+ + V(\tau,T)\big) \end{bmatrix}
\end{aligned}$$

Now, realising that we can combine two terms since $V(t,\tau) + V(\tau,T) \equiv V(t,T)$ we have:

$$\tilde{V}(t,T) = E^Q\begin{bmatrix} I(\tau > T)V(t,T) + \\ I(\tau \le T)V(t,T) + \\ I(\tau \le T)\big((R-1)V(\tau,T)^+\big) \end{bmatrix}.$$

Finally, since $I(\tau > T)V(t,T) + I(\tau \le T)V(t,T) \equiv V(t,T)$, we have:

$$\tilde{V}(t,T) = V(t,T) - E^Q\left[(1-R)I(\tau \le T)V(\tau,T)^+\right].$$

The above equation is crucial since it defines the risky value of a netting set of derivatives positions with respect to the risk-free value. The relevant term is often known as CVA (credit value adjustment). It is an adjustment to the risk-free value of the positions within the netting set to account for counterparty risk:

$$\tilde{V}(t,T) = V(t,T) - CVA(t,T).$$

$$CVA(t,T) = E^Q\left[(1-R)I(\tau \le T)V(\tau,T)^+\right].$$

Note that we made the assumption that the future MtM value, $V(s,T)$, includes discounting for notational simplicity. If we drop this assumption, the above formula will include discounting:

$$CVA(t,T) = E^Q\left[(1-R)I(\tau \le T)V(\tau,T)^+\frac{\beta(t)}{\beta(\tau)}\right],$$

where $\beta(s)$ is the value of the "money market account" at time s.

APPENDIX 14B: CVA computation details

To derive the classic CVA formula we can then write:

$$CVA(t,T) = -(1-\bar{R})E^Q\left[I(u \le T)V^*(u,T)^+\right],$$

Where $\bar{R}$ is the mean or expected recovery value. We use $V^*(u,T)$ to denote:

$$V^*(u,T) = V(u,T)|\tau = u.$$

This is a critical point in the analysis as the above statement requires the exposure at a future date, $V(u,T)$, *knowing* that default of the counterparty has occurred at that date $(\tau = u)$. Ignoring wrong-way risk simple sets $V^*(u,T) = V(u,T)$ which we do from now on.

Since the expectation in the above equation is over all times before the final maturity, we can integrate over all possible default times. We obtain:

$$CVA(t,T) = -(1-\bar{R})E^Q\left[\int_t^T B(t,u)V(u,T)^+ dF(t,u)\right],$$

where $B(t,u)$ is the risk-free discount factor and $F(t,u)$ is the cumulative default probability for the counterparty (probability of no default) as described in Appendix 12B.

We recognise the discounted expected exposure calculated under the risk-neutral measure denoted by $EE_d(u,T) = E^Q\left[B(t,u)V(u,T)^+\right]$. Assuming that the default probabilities are deterministic, we have:

$$CVA(t,T) = (1-\bar{R})\left[\int_t^T EE_d(u,T)dF(t,u)\right],$$

Finally, we could compute the above equation via some integration scheme such as:

$$CVA(t,T) \approx -(1-\bar{R})\sum_{i=1}^{m} EE_d(t_i,T)[F(t,t_i) - F(t,t_{i-1})],$$

where we have m periods given by $[t_0(=t), t_1, ..., t_m(=T)]$. As long as m is reasonably large then this will be a good approximation.

With further simplifying assumptions, one can obtain a simple expression for CVA linked to the credit spread of the counterparty. To do this we have to work with the non-discounted expected exposure[1] as therefore start from the following CVA formula:

$$CVA(t,T) = -(1-\bar{R})\left[\int_t^T EE(u,T)B(t,u)dF(t,u)\right]$$

[1] As described in Chapter 14, this can be done if EE is computed using the T-forward measure.

APPENDIX 14C: Approximate CVA formula

Suppose that we approximate the (undiscounted) expected exposure term, $EE(u,T)$, as a fixed known amount which would obviously be the EPE. The fixed EPE (defined in Chapter 8) would most obviously be computed from the EE averaged over time, for example:

$$EPE = \frac{\int_t^T EE(u,T)du}{T-t} \approx \frac{1}{m}\sum_{j=1}^{m} EE(t,t_j)$$

Clearly, the approximation will be a good one if the relationship between EPE, default probability (and indeed discount factors) is reasonably homogeneous through time or other cancellation effects come into play. Using this approach, the CVA is:

$$CVA(t,T) \approx (1-\overline{R})E^Q\left[\int_t^T B(t,u)dF(t,u)\right]EPE$$

.

Recalling Appendix 12B, we can see this is simply the value of CDS protection on a notional equally the EPE. Hence we have the following approximation giving a running CVA (i.e. expressed as a spread):

$$CVA \approx EPE \times Spread \text{ .}$$

We can use the risky annuity formulas (Appendix 12B) to convert this to an up-front value.

APPENDIX 14E: Incremental CVA formula

To calculate incremental CVA, we need to quantify the change before and after added a new trade, *i*:

$$
\begin{aligned}
& CVA^{NS+i}(t,T) - CVA^{NS}(t,T) \\
&= (1-\overline{R})\sum_{i=1}^{m} EE^{NS+i}(t,t_i)\left[F(t,t_i)-F(t,t_{i-1})\right] - (1-\overline{R})\sum_{i=1}^{m} EE^{NS}(t,t_i)\left[F(t,t_i)-F(t,t_{i-1})\right] \\
&= (1-\overline{R})\sum_{i=1}^{m} \left[EE^{NS+i}(t,t_i) - EE^{NS}(t,t_i)\right]\left[F(t,t_i)-F(t,t_{i-1})\right]
\end{aligned}
$$

We therefore simply need to use the incremental EE, $EE^{NS+i}(t,t_i) - EE^{NS}(t,t_i)$ in the standard CVA formula.

APPENDIX 14F: Deriving the bilateral CVA formula.

We wish to find an expression for the risky value, $\widetilde{V}(t,T)$, of a netted set of derivatives positions with a maximum maturity date T as in Appendix 12A but under the assumption that the institution concerned may also default in addition to their counterparty. Denoting the default time of the institution as τ_I, their recovery value as R_I and following the notation and logic in Appendix 12A, we now have the following cases (we denote the "first-to-default time" of the institution and counterparty as $\tau^1 = \min(\tau, \tau_I)$).

1) ***Neither counterparty nor institution defaults before*** T

In this case, the risky position is equivalent to the risk-free position and we write the corresponding payoff as:

$$I(\tau^1 > T)V(t,T).$$

2) ***Counterparty defaults first and also before time*** T **.**

This is the default payoff as in Appendix 7.A:

$$I(\tau^1 \leq T)I(\tau^1 = \tau)\left(RV(\tau^1,T)^+ + V(\tau^1,T)^-\right).$$

3) ***Institution defaults first and also before time*** T

This is an additional term compared with the unilateral CVA case and corresponds to the institution itself defaulting. If they owe money to their counterparty (negative MtM) then they will pay only a recovery fraction of this whilst if the counterparty owes them money (positive MtM) then they will still receive this. Hence, the payoff is the opposite of case 2 above:

$$I(\tau^1 \leq T)I(\tau^1 = \tau_I)\left(R_I V(\tau_I,T)^- + V(\tau_I,T)^+\right).$$

4) ***If either the institution or counterparty does default then all cashflows prior to the first-to-default date will be paid***

$$I(\tau^1 \leq T)V(t,\tau^1).$$

Putting the above payoffs together, we have the following expression for the value of the risky position:

$$\widetilde{V}(t,T) = E^Q\begin{bmatrix} I(\tau^1 > T)V(t,T) + \\ I(\tau^1 \leq T)V(t,\tau^1) + \\ I(\tau^1 \leq T)I(\tau^1 = \tau)\left(RV(\tau^1,T)^+ + V(\tau^1,T)^-\right) + \\ I(\tau^1 \leq T)I(\tau^1 = \tau_I)\left(R_I V(\tau^1,T)^- + V(\tau^1,T)^+\right) \end{bmatrix}.$$

Similarly to Appendix 7.A, we simplify the above expression as:

$$\tilde{V}(t,T) = E^Q \begin{bmatrix} I(\tau^1 \le T)V(t,T) + \\ I(\tau^1 \le T)V(t,\tau^1) + I(\tau^1 = \tau)V(\tau^1,T) + I(\tau^1 = \tau_I)V(\tau^1,T) + \\ I(\tau^1 \le T)I(\tau^1 = \tau)\left(RV(\tau^1,T)^+ - V(\tau^1,T)^+\right) + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)\left(R_I V(\tau^1,T)^- - V(\tau^1,T)^-\right) \end{bmatrix}$$

Finally obtaining:

$$\tilde{V}(t,T) = V(t,T) + E^Q \begin{bmatrix} I(\tau^1 \le T)I(\tau^1 = \tau)\left(RV(\tau^1,T)^+ - V(\tau^1,T)^+\right) + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)\left(R_I V(\tau^1,T)^- - V(\tau^1,T)^-\right) \end{bmatrix}.$$

$$\tilde{V}(t,T) = V(t,T) - E^Q \begin{bmatrix} I(\tau^1 \le T)I(\tau^1 = \tau)(1-R)V(\tau^1,T)^+ + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)(1-R_I)V(\tau^1,T)^- \end{bmatrix}.$$

We can identify the BCVA (bilateral CVA) term as being:

$$BCVA(t,T) = E^Q \begin{bmatrix} I(\tau^1 \le T)I(\tau^1 = \tau)(1-R)V(\tau^1,T)^+ + \\ I(\tau^1 \le T)I(\tau^1 = \tau_I)(1-R_I)V(\tau^1,T)^- \end{bmatrix}.$$

Finally, under the similar assumptions of no wrong-way risk and of no simultaneous default between the default of the institution and its counterparty, we would have a formula analogous to that derived in Appendix 12B for computing BCVA:

$$\begin{aligned} & BCVA(t,T) = \\ & -(1-\overline{R})E^Q\left[\int_t^T B(t,u)V(u,T)^+ S_I(u)dS(t,u)\right] \; . \\ & +(1-R_I)E^Q\left[\int_t^T B(t,u)V(u,T)^- S(u)dS_I(t,u)\right] \end{aligned}$$

An obvious approximation to compute this formula using the discounting EE and NEE would then be:

$$\begin{aligned} & BCVA(t,T) \approx (1-\overline{R})\sum_{i=1}^{m} EE_d(t,t_i)S_I(t,t_{i-1})\left[F(t,t_i) - F(t,t_{i-1})\right] \\ & -(1-\overline{R}_I)\sum_{i=1}^{m} NEE_d(t,t_i)S(t,t_{i-1})\left[F_I(t,t_i) - F_I(t,t_{i-1})\right] \end{aligned} .$$

More details on these calculations and discussion on incorporating dependency between the default of the institution and the counterparty can be found in Gregory (2009a).

A simple spread based approximation would be:

$$CVA \approx EPE \times Spread - ENE \times Spread_I ,$$

where $Spread_I$ represents the credit spread of the institution themselves.